Exploring Magnetic Monopoles in D=4 (Part 1)

Write Your Name Here:

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1 Electric–Magnetic Duality.

Consider Maxwell’s equations:

\[
\nabla \cdot E = \rho_e ,
\]
\[
\nabla \times B - \frac{\partial E}{\partial t} = j_e ,
\]
\[
\nabla \cdot B = 0 ,
\]
\[
\nabla \times E + \frac{\partial B}{\partial t} = 0 .
\]

(1) Remind yourself of the covariant form of these equations. Write them down.

(2) Imagine that there are point sources of magnetic charge. Write down explicitly modified Maxwell’s equations to include these sources.

(3) Hence deduce a form of electric/magnetic duality of the equations, which exchanges electric and magnetic quantities. Write down the transformation.

(4) Deduce (perhaps after some discussion with your neighbours, or recollection from the lectures) the covariant form of the modified Maxwell equations, and also express the form of the duality transformation in this covariant language.

2 Monopoles and Angular Momentum

A well-known (I hope) solution is for a point source of electric charge \( q \) located at the origin:

\[
\rho_e = 4\pi q \delta(x) , \quad j_e = 0 , \quad \rho_m = 0 , \quad j_m = 0 ,
\]

(5)
with fields:
\[ \mathbf{E} = \frac{q}{|\mathbf{x}|^3} \mathbf{x}, \quad \mathbf{B} = 0 \, . \] (6)

(1) Write down an analogous solution for a point magnetic charge, denoting the charge by \( g \).

(2) What is the equation of motion for a point particle of electric charge \( q \) and mass \( m \) moving in a magnetic field? Perhaps the words \textit{Lorentz Force} might help.

(3) Prove that, in the \( \mathbf{B} \) field produced by our point magnetic source, and using your expression for the equation of motion, that the orbital angular momentum \( \mathbf{L} = \mathbf{x} \times \mathbf{p} \) is not conserved.

\textit{Hint: use} \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \)

(4) Using the failure of conservation of \( \mathbf{L} \), show that this operator is conserved:
\[ \mathbf{J} = \mathbf{L} - gq \frac{\mathbf{x}}{|\mathbf{x}|} \, . \] (7)

(5) Quantum mechanics actually sets the eigenvalues of \( \mathbf{J} \) to be half–integers. Deduce therefore the Dirac relation that sets the basic unit of an electric charge quantum:
\[ q = \frac{n}{2g} \, , \] (8)

where \( n \in \mathbb{Z} \).

3\quad \textbf{Closer Look at the Quantum Mechanics.}

Recall (I hope!) Schrödinger’s equation:
\[ i \frac{\partial \psi}{\partial t} = -\frac{\nabla^2}{2m} \psi \, . \] (9)

(1) Introduce the vector and scalar potentials for electromagnetism. Write their definitions.

(2) Recall how Schrödinger’s equation is modified in the presence of electromagnetism.

(3) A gauge transformation on \( \mathbf{A} \) is defined here as:
\[ \mathbf{A}(\mathbf{x}, t) \rightarrow \mathbf{A}(\mathbf{x}, t) + \frac{1}{q} \nabla \omega(\mathbf{x}) \, , \] (10)

for a static change of gauge. Show that Schrödinger’s equation is invariant only if you include a transformation on the wavefunction. Deduce the precise form.

In Cartesian coordinates, we can write a potential for a magnetic point source as follows:
\[ A_1^N = -\frac{x_2}{|\mathbf{x}|(|\mathbf{x}| + x_3)} g \, , \] (11)
\[ A_2^N = \frac{x_1}{|\mathbf{x}|(|\mathbf{x}| + x_3)} g \, , \] (12)
\[ A_3^N = 0 \, . \] (13)
(4) Study this configuration, and how it gives the $\mathbf{B}$ field. Can you say anything interesting about the negative $x_3$ axis that follows from Stokes’ Theorem? What?

(5) Show that in spherical polars, $(r, \theta, \phi)$, where $\phi$ is the polar angle, that the configuration above is:

$$A_r^N = 0, \quad A_\phi^N = \frac{g}{r} \frac{1 - \cos \theta}{\sin \theta}, \quad A_\theta^N = 0.$$ (14)

(6) Write analogous expressions for $A_i^S$ in polars.

(7) To make contact with our expressions in the lectures, we can transform to the language of forms. $A$ is a one–form:

$$A = A_i dx^i.$$ (15)

Examine the expressions above and deduce the tangent space basis for spherical polars:

$$dr, \quad r \sin \theta d\phi, \quad r d\theta.$$ (16)

(8) Hence verify that we can write

$$A^{N(S)} = g(1 \pm \cos \theta) d\phi.$$ (17)

(9) By asking for the difference between the N and S patches of the two–sphere to be a gauge transformation

$$A^N - A^S = \frac{i}{q} \Omega d\Omega^{-1}, \quad \Omega = e^{i\omega},$$ (18)

deduce (as we did in the lectures) Dirac’s quantisation condition on the electric charge by requiring the wavefunction to be single–valued as one goes around (say) the equator.