

# Exploring Magnetic Monopoles in D=4 (Part 3)

Write Your Name Directly Below:

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## 1 Symmetry Breaking

Consider The following Lagrangian for the  $SO(3)$  Yang–Mills–Higgs system:

$$\mathcal{L} = -\frac{1}{4}\mathbf{F}_{\mu\nu} \cdot \mathbf{F}^{\mu\nu} + \frac{1}{2}D^\mu\phi \cdot D_\mu\phi - V(\phi) . \quad (1)$$

Here,  $\mathbf{A}_\mu = (A_\mu^a)$ , and  $\phi = (\phi^a)$ , with  $\mu, \nu, \sigma = 0, 1, 2, 3$ , and  $a, b, c = 1, 2, 3$ , etc. The latter are  $SO(3)$  indices and the dot product is in this “isospin” space. Sometimes a cross product will also appear, for example in:

$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu - e\mathbf{A}_\mu \times \mathbf{A}_\nu , \quad D_\mu\phi = \partial_\mu\phi + e\mathbf{A}_\mu \times \phi . \quad (2)$$

Also, note that:

$$V(\phi) = \frac{\lambda}{4}(|\phi|^2 - v^2)^2 . \quad (3)$$

(1) Sketch the potential. Discuss what happens to the gauge symmetry when a minimum of the potential is selected. Discuss the key features of the spectrum of fields before and after the symmetry breaking.

(2) By deriving the Hamiltonian, prove that the mass–energy is:

$$M = \int d^3x \left\{ \frac{1}{4}\mathbf{F}_{ij} \cdot \mathbf{F}^{ij} + \frac{1}{2}D_i\phi \cdot D^i\phi + V(\phi) \right\} . \quad (4)$$

We have used indices  $i, j, k = 1, 2, 3$  for the spatial coordinates.

We will consider a configuration such that as  $r \rightarrow \infty$ , has

$$\phi^a \rightarrow v \frac{x^a}{r} . \quad (5)$$

(2) Discuss why, for a finite energy solution, we also need to consider a gauge field which in the limit  $r \rightarrow \infty$ , goes as:

$$A_0 = 0 , \quad A_i^b \rightarrow \frac{1}{e} \epsilon^{bij} \frac{x^j}{r^2} . \quad (6)$$

It turns out that the electromagnetic field strength for the theory is:

$$\mathcal{F}_{\mu\nu} = \frac{1}{|\phi|} \phi \cdot \mathbf{F}_{\mu\nu} - \frac{1}{e|\phi|^3} \phi \cdot (D_\mu \phi \times D_\nu \phi) . \quad (7)$$

(3) Explain why there no electric charge for our configuration. What modification could you make to give our solution electric charge without destroying the behaviour in equation (5)?

Define the electromagnetic vector potential as

$$A_\mu = \frac{1}{|\phi|} \phi \cdot \mathbf{A}_\mu . \quad (8)$$

From this, we get:

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{e|\phi|^3} \epsilon_{abc} \phi^a \partial_\mu \phi^b \partial_\nu \phi^c . \quad (9)$$

(4) Show that our ansatz above gives a configuration with a radial magnetic field, and write down (given what we did in the previous worksheet) the magnetic flux associated with it.

## 2 Bogomol'nyi–Prasad–Sommerfeld Bound

(1) Using that:

$$A^2 + B^2 = (A \pm B)^2 \mp 2AB , \quad (10)$$

Prove that

$$M \geq \int d^3x \left\{ \mp \frac{1}{2} \epsilon_{ijk} \mathbf{F}_{ij} \cdot D_k \phi + V(\phi) \right\} . \quad (11)$$

*Hint: Use your experience from the kink solution's Bogomol'nyi bound studied in worksheet 1.*

### 3 Magnetic Charge and Topology

Let us define the magnetic current as we did two worksheets ago:

$$j_{\mathbf{m}}^{\mu} = \partial_{\nu} {}^*F^{\mu\nu} , \quad (12)$$

where

$${}^*F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} . \quad (13)$$

(1) Show that

$$j_{\mathbf{m}}^{\mu} = -\frac{1}{2e} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abc} \partial_{\nu} \hat{\phi}^a \partial_{\rho} \hat{\phi}^b \partial_{\sigma} \hat{\phi}^c . \quad (14)$$

(2) Show that this current is conserved.

(3) Write an integral expression for the magnetic charge  $g$ . From your expression, and perhaps with a little in-class discussion, write in your own words how we arrive at the result that the charge is quantized as:

$$g = \frac{n}{e} , \quad n \in \mathbb{Z} . \quad (15)$$

(4) Discuss the connection of this with the magnetic charge you deduced in the previous section.

(5) Discuss the comparison of this charge condition with the one derived for the Dirac monopole.  $g$  is twice as big as it need be. To what might this be traceable? (What sets the normalisations of charges in a gauge theory?)