PHYS 3622 Modern Laboratory Methods II

Coaxial Transmission Lines

Purpose

Electromagnetic waves are the means by which much of the information that forms a part of daily life is transmitted. This information may be in the form of a telephone call, radio and television broadcasts, digital signals inside a computer, etc. In this experiment, you will examine the propagation of transient and steady state signals in a transmission line.

Equipment

- Pulse generator
- Sine-wave oscillator
- Oscilloscope
- Transmission line
- BNC tees
- Line terminators
  - Short-circuit
  - 4.7 Ω
  - 47 Ω
  - 470 Ω

Background

The Characteristic Impedance of Coaxial Transmission Lines*

Figure 1 shows a voltage source, V, connected to a load impedance Z_L by a coaxial cable. If the source is a DC source, a current, I, flows down the center conductor, through the load, and back to the source via the outer conductor. Elementary electromagnetic theory states that there are corresponding **E** and **H** fields (as shown in the figure) inside the cable, and that there are no electromagnetic fields outside of the cable. For low-frequency AC sources, the description is essentially the same as for DC. However, at higher frequencies, where the wavelength, λ, is comparable to the length, ℓ, of the transmission line, it is useful to describe the problem in terms of electromagnetic waves traveling on the line.

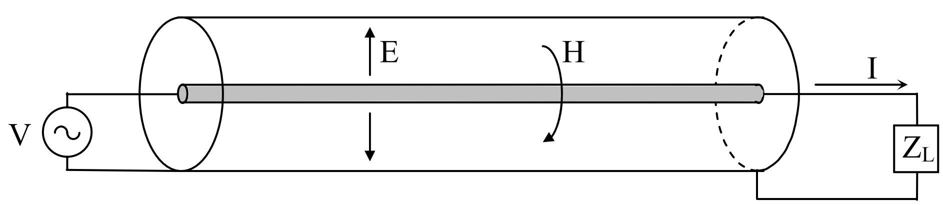


Figure 1. A coaxial transmission line terminated by a load impedance Z_L

In order to determine the fields in the coaxial transmission line, we would solve Laplace's equation, then, from the solution, determine the vectors **E** and **H**. We would then find that the magnitudes of the vectors are related by $E=Z_oH$, where Z_o is the characteristic impedance of the transmission line. An alternative approach will be to solve for the electric potential difference (voltage) between the conductors and the current in the line directly from an electrical equivalent circuit. This is the approach we will take in this experiment in order to determine the characteristic impedance of the line and also to look at reflection in the line at the load.

Figure 2 shows a cross section of a coaxial transmission line. The line consists of two conductors, having radii shown in the figure, separated by a dielectric insulator. When determining the electrical equivalent circuit of the transmission line we must take into account

* References: *The Electromagnetic Field* by Shadowitz and *Electromagnetic Fields* by Wangsness

the resistive, capacitive, and inductive impedances of the line. The resistive impedances consist of the resistance of the metallic conductors and the resistance of the insulating material. The outer conductor is typically at a different potential than the inner conductor (the outer conductor is typically at “ground” potential) and any voltage on the line is impressed on the inner conductor. As a consequence of this, the line can be thought of as a cylindrical capacitor with the conductors separated by the insulating material. And since, as Figure 1 indicates, there is a magnetic field associated with electromagnetic waves traveling on the transmission line, there will also be an associated inductance. It is difficult to lump any impedance into a single term since a total for each is dependent on the length of the line. So the best approach is to write each term in per unit length units. Thus by taking a small section of the line and denoting the resistive, inductive, and capacitive properties as per unit length an electrical equivalent circuit can be drawn as in Figure 3. Note that G is the conductance of the material that electrically insulates the center conductor from the outer conductor. The conductance is equal to $1/R_i$, where R_i is the resistance of the insulating material.

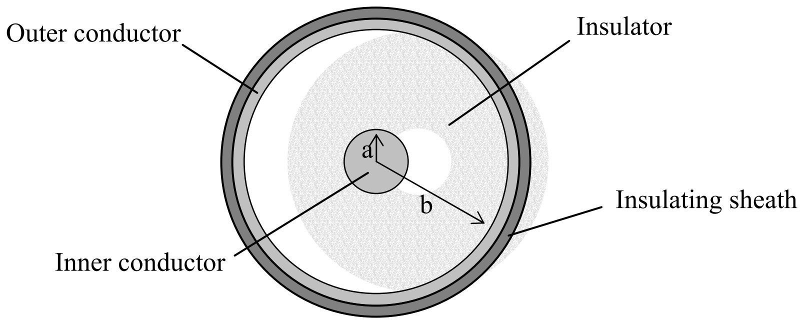
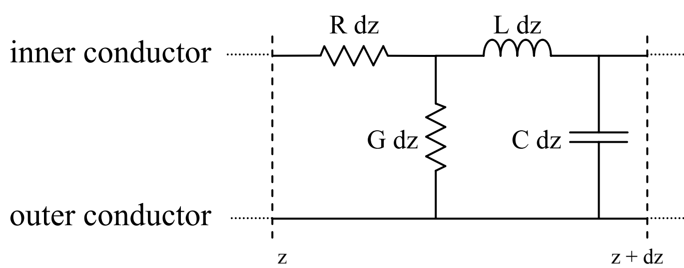


Figure 2. Cross-section of a coaxial transmission line.

Figure 3. The equivalent electrical circuit of a small section of coaxial transmission line. The impedances shown are in per length units, thus R has units of Ω/m.



With this approach, we look to find the time varying voltage and current (we are looking at the AC case) and in order to do this, we seek solutions to the voltage and current waves traveling in the transmission line of the form

$$V(z,t) = V(z)\, e^{i\omega t} \tag{1}$$

$$I(z,t) = I(z)\, e^{i\omega t} \quad . \tag{2}$$

We can start by taking a section of the transmission line that extends from z to z+dz (Figure 3). If I(z) is the current at z then the current at z+dz is less than I(z) by the current across the line (Though R_i is very large it is still finite and consequently there may be some leakage across the conductors). We can write this difference in current as

$$I(z) - I(z+dz) = \left(\frac{1}{R_i} + i\omega C\right) V(z)\, dz = \left(G + i\omega C\right) V(z)\, dz \tag{4}$$

or

$$\frac{dI(z)}{dz} = -(G + i\omega C)\, V(z) \tag{5}$$

Note that V is the voltage across the line, between the two conductors. Taking a loop through the dashed lines of Figure 3 (Kirchhoff's Loop Rule), we get

$$V(z+dz) - V(z) + (R + i\omega L)\, I(z)\, dz = 0 \tag{6}$$

or,

$$\frac{dV(z)}{dz} = -(R + i\omega L)\, I(z) \quad . \tag{7}$$

From Equations 5 and 7 we see that V and I are related so we can solve for one and determine the other easily enough. Differentiating Equation 7 with respect to z and inserting Equation 5 yields

$$\frac{\partial^2 V(z)}{\partial z^2} = \gamma^2\, V(z) \quad . \tag{8}$$

where

$$\gamma = \left[(G + i\omega C)(R + i\omega L) \right]^{1/2} \tag{9}$$

The second order differential equation of Equation 8 has the general solution

$$V(z) = V_1 e^{-\gamma z} + V_2 e^{\gamma z} \quad . \tag{10}$$

This solution can then be used to determine I(z). Thus, differentiating Equation 10 and using Equation 7 to solve for I(z) yields

$$I(z) = \left(\frac{G + i\omega C}{R + i\omega L} \right)^{1/2} \left[V_1\, e^{-\gamma z} - V_2\, e^{\gamma z} \right] \tag{11}$$

The term in front of the brackets of Equation 11 is the reciprocal of an impedance term (recall Ohm's Law), where the impedance is the impedance at any point on the transmission line. Whereas R, L, G, and C have units of per unit length, this impedance term, usually called the characteristic impedance of the line, has units of ohms. So let

$$Z_o = \left(\frac{R + i\omega L}{G + i\omega C} \right)^{1/2} \quad . \tag{12}$$

In general, γ (Equation 9) is a complex number, that is, it is comprised of both an imaginary term and a real term, so let

$$\gamma = \alpha + i\,\beta \quad . \tag{13}$$

Specifically, α and β are (recall that R, G, L, and C are per unit length),

$$\alpha = \frac{1}{\sqrt{2}} \left\{ \left(RG + \omega^2 LC \right) + \left[\left(R^2 + \omega^2 L^2 \right)\left(G^2 + \omega^2 C^2 \right) \right]^{1/2} \right\}^{1/2} , \tag{14}$$

$$\beta = \frac{1}{\sqrt{2}} \left\{ -\left(RG + \omega^2 LC \right) + \left[\left(R^2 + \omega^2 L^2 \right)\left(G^2 + \omega^2 C^2 \right) \right]^{1/2} \right\}^{1/2} \quad . \tag{15}$$

With the general solutions given by Equations 1 and 2, and with Equations 10 and 11, the complete general solutions for waves traveling on the transmission line represented by Figure 1 are

$$V(z,t) = V_1\, e^{-\alpha z}\, e^{i(\omega t - \beta z)} + V_2\, e^{\alpha z}\, e^{i(\omega t + \beta z)} \tag{16}$$

and

$$I(z,t) = \frac{1}{Z_o}\left(V_1\, e^{-\alpha z}\, e^{i(\omega t - \beta z)} - V_2\, e^{\alpha z}\, e^{i(\omega t + \beta z)}\right) \quad . \tag{17}$$

The first terms in V(z,t), Equation 16, and I(z,t), Equation 17, represent waves traveling to the right with speed $v = \omega/\beta$ and wavelength $\lambda = 2\pi/\beta$ that are damped by the factor $e^{-\alpha z}$. The second terms in V(z,t) and I(z,t) represent waves traveling to the left with the same speed and wavelength as the right-going waves, but damped by the factor $e^{\alpha z}$. For waves traveling along the transmission line, the speed is frequency dependent and thus is an example of a dispersive medium. Hence, a wave pulse comprised of a number of different frequencies will tend to spread out spatially, with the lower frequency waves following the higher frequency waves.

For the case of high frequencies (for coaxial transmission lines, a high frequency means all frequencies above, say, 100 kHz) and low losses, the variables α and β can be approximated in order to simply them and to get an idea of the magnitude of each. The high frequency and low loss approximations are

$$\frac{\omega L}{R} >> 1 \tag{18}$$

and

$$\frac{\omega C}{G} >> 1 \quad . \tag{19}$$

Equation 18 represents the low loss due to waves traveling down the line, in that R, the resistance of the inner conductor is small since it usually metallic. Equation 19 represents a low loss condition due to the very high resistance R_i ($G \approx 0$) of the insulating material that separates the inner and outer conductors. With these approximations then the following is true,

$$Z_o \approx \sqrt{\frac{L}{C}} \quad , \quad \alpha \approx \frac{R}{2}\sqrt{\frac{C}{L}} \quad \text{and} \quad \beta \approx \omega\sqrt{LC} \quad . \tag{20}$$

Note that

$$\frac{\alpha}{\beta} \approx \frac{R}{2\omega L} << 1 \ , \tag{21}$$

which satisfies the imposed condition represented by Equation 18 and states that $\alpha << \beta$. Recall that α is the real term of γ and determines the damping of the waves. In the high frequency regime, α is directly proportional to R, and since R, the resistance of the metallic conductor, is very small, we are free to set α equal to zero. It should also be noted that in the high frequency approximation, Z_o is a real quantity and that the speed of the waves is given by

$$v = \frac{1}{\sqrt{LC}} \quad . \tag{22}$$

In the high frequency limit, Equation 22 implies that the wave speed is independent of frequency, that is, no dispersion, and is the same as if the line had no resistance.

For a coaxial transmission line, the capacitance per unit length and the inductance per unit length are given by,

$$\frac{C}{\ell} = \frac{2\pi\varepsilon}{\ln(b/a)} \quad \text{and} \quad \frac{L}{\ell} = \frac{\mu}{2\pi}\ln(b/a) \;, \tag{23}$$

where ε and μ are the permittivity and permeability, respectively, of the material insulating the inner conductor from the outer conductor of the coaxial transmission line. The values "b" and "a" are the radii of the outer and inner conductors, respectively, of the coaxial transmission line and ℓ is the length of the transmission line. The permeability μ is typically that of the permeability of free space μ_o and has the value $4\pi \times 10^{-7}\ \mathrm{H/m}$. The permittivity ε differs from that of the free space value, ε_o, but can be written $k\varepsilon_o$, where k is the dielectric constant of the insulating material and $\varepsilon_o = 8.85 \times 10^{-12}\ \mathrm{F/m}$. Thus Z_o (Equation 20) and the wave speed (Equation 22) can be written as,

$$Z_o = \frac{60\ \Omega}{k^{1/2}}\ln\left(\frac{b}{a}\right) \tag{24}$$

and

$$v = \frac{c}{k^{1/2}} \tag{25}$$

where c is the speed of light in vacuum. In the high frequency limit, the characteristic impedance of the coaxial transmission line Z_o is determined by both the geometry of the conductor and the dielectric constant of the insulating material and the speed of the waves is determined by the dielectric constant. Consequently, a direct measurement of the speed that waves travel on the transmission line will determine the dielectric constant and by a direct measurement of "a" and "b" the characteristic impedance of the transmission line can be found.

Reflection In Transmission Lines Terminated By A Real Impedance

The solutions given by Equations 16 and 17 are for waves traveling on a line of infinite length. Now we will deal with the situation where the line is terminated by some load impedance Z_L (cf Figure 1) for the cases $Z_L = Z_o$ and $Z_L \neq Z_o$.

For the case where the line is of finite length then the load Z_L on the line determines the magnitude of the unknown coefficients V_1 and V_2. In general, Z_L can be real (a resistor) or expressed as complex (an inductor, capacitor or any combination thereof that includes a resistor) and though much of the analysis to come is true for complex Z_L, we will consider only the case Z_L real.

In the approximation of high frequencies (α=0) V(z,t) and I(z,t) are

$$V(z,t) = V_1\ e^{i(\omega t - \beta z)} + V_2\ e^{i(\omega t + \beta z)} \tag{26}$$

and

$$I(z,t) = \frac{1}{Z_o}\left(V_1\ e^{i(\omega t - \beta z)} - V_2\ e^{i(\omega t + \beta z)}\right) \;. \tag{27}$$

A wave just leaving the source will have its voltage and current characteristics of the form given by the first term on the right side of Equations 26 and 27. At all times as this initial wave is traveling to the right, with reference to direction given by Figure 1, before it reaches the load, $V_1/I_1=Z_o$. That is Ohm's Law is satisfied. When these initial waves reach the load two things can happen:

1) If $Z_L = Z_o$ then the ratio $V_1/I_1=Z_o=Z_L$, and the wave travels through the load without attenuation; Ohm's Law is satisfied for this load impedance.

2) If $Z_L \neq Z_o$ then the ratio $V_1/I_1 \neq Z_L$, that is, the ratio does not have the correct value to satisfy Ohm's Law for the load impedance. In order to satisfy Ohm's Law at the load there must be an instantaneous change in the value of V_1 and I_1 at the load. Thus, part of the waves travel through the load and the instantaneous change or discontinuity in the values of V_2 and I_2 then propagates to the left, toward the source. The wave that travels to the left can then be reflected at the source and travel back to the load where it can then be reflected once again. When $Z_L \neq Z_o$ an infinite number of reflections can occur at the load and after a short time a steady-state condition is approached in which the amplitudes of the voltage and current waves approach the values required by Ohm's Law at the load. In all, steady state standing waves form on the line by the waves produced by the AC source and the waves reflected at the load. Note that the final, steady state, V_1 and I_1 values of the right going waves will differ from the initial values of V_1 and I_1 of the initial waves. That is to say that if the initial voltage wave had an amplitude of 10 V then the final voltage wave will have a different amplitude due to the reflections that have traversed the length of the line and returned to the load.

Keeping the general term γ for now, we can simplify further analysis by considering only the spatial part of V(z,t) and I(z,t),

$$V(z) = V_1\, e^{-\gamma z} + V_2\, e^{\gamma z} \qquad (28)$$

and

$$I(z) = I_1 e^{-\gamma z} - I_2 e^{\gamma z} \;, \qquad (29)$$

where $I_2=V_2/Z_o$ and $I_1=V_1/Z_o$, and give reference to Figure 4 for the variables z and x (to be used later).

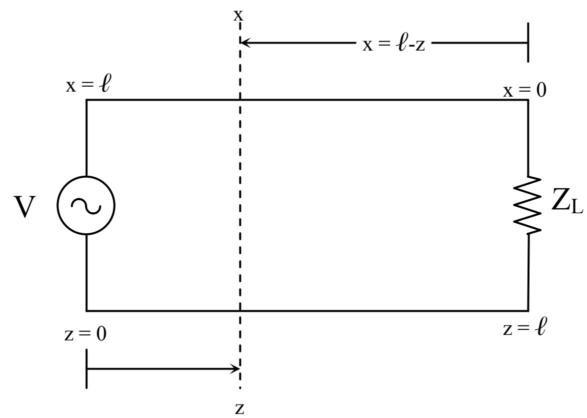


Figure 4.

At the load end ($z = \ell$),

$$Z_L = \frac{V_1\, e^{-\gamma\ell} + V_2\, e^{\gamma\ell}}{\frac{V_1}{Z_o}\, e^{-\gamma\ell} - \frac{V_2}{Z_o}\, e^{\gamma\ell}} \qquad (30)$$

We can define a reflection coefficient Γ_L that is the ratio of the steady state reflected wave to the steady state incident wave. Thus,

$$\Gamma_L = \frac{V_2 e^{\gamma\ell}}{V_1 e^{-\gamma\ell}} = \frac{V_2}{V_1} e^{2\gamma\ell} \qquad (31)$$

In terms of Γ_L, Z_L becomes

$$Z_L = Z_o \frac{1+\Gamma_L}{1-\Gamma_L} \qquad (32)$$

Solving this for Γ_L,

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} \qquad (33)$$

Though we have stated that we are dealing with real values of Z_L, that is purely resistive loads, this expression (Equation 33) is a general expression for all load impedances.

Γ_L depends on the difference between the load impedance and the impedance of the line. Γ_L can be positive or negative, and have a magnitude that ranges from zero to one. We can consider several cases that will encompass all the physics:

1) When $Z_L = Z_o$ then $\Gamma_L = 0$. In this case all the power from the source is delivered to the load. There is no reflected wave.
2) When $Z_L = \infty$, the line is not terminated (it is an open circuit) and Γ_L=1. All incident waves are reflected at the end of the line. The reflected wave has the same amplitude as the incident wave $\left(|\Gamma_L| = 1 \right)$ and there is no change in phase upon reflection (Γ_L is positive).
3) When $Z_L = 0$ the end of the line is short-circuited and $\Gamma_L = -1$. The reflected wave has the same amplitude as the incident wave $\left(|\Gamma_L| = 1 \right)$ and there is a 180° phase change upon reflection (Γ_L is negative).
4) When $Z_o < Z_L < \infty$ then $0 < \Gamma_L < 1$. The amplitude of the reflected wave is a smaller than the amplitude of the incident wave $\left(|\Gamma_L| < 1 \right)$ and there is no change in phase upon reflection (Γ_L is positive).
5) When $0 < Z_L < Z_o$ then $-1 < \Gamma_L < 0$. The amplitude of the reflected wave is a smaller than the amplitude of the incident wave $\left(|\Gamma_L| < 1 \right)$ and there is a 180° phase change upon reflection (Γ_L is negative).

We are primarily concerned with what happens at the load and write our expressions for the steady state voltage and current waves accordingly. So switching from variable z to variable x as depicted in Figure 4, we get, with $\gamma = i\,\beta$ (that is α=0) and Equation 33,

$$\begin{aligned} V(x) &= V_1\, e^{-i\beta\ell} e^{\beta x} + V_2\, e^{i\beta\ell} e^{-i\beta x} \\ &= \frac{2\, V_1\, e^{-i\beta\ell}}{Z_L + Z_o}\left[Z_L \cos\beta x + i\, Z_o \sin\beta x\right] \end{aligned} \tag{34}$$

and similarly,

$$I(x) = \frac{2\, I_1\, e^{-i\beta\ell}}{Z_L + Z_o}\left(Z_L \sin\beta x + i\, Z_o \cos\beta x\right) \quad . \tag{35}$$

V(x) and I(x) are complex expressions and it can be seen that there is a phase difference that depends on Z_L. Since it is the magnitude of the voltage that can be measured with a voltmeter (you can't measure phase with a voltmeter) we will concern ourselves with only the magnitudes of the steady state waves. Actually we are only interested in the steady state voltage but the current will be included for now. Thus,

$$\begin{aligned} |V| &= \frac{2\, V_1}{Z_L + Z_o}\sqrt{\left(Z_L \cos\beta x\right)^2 + \left(Z_o \sin\beta x\right)^2} \\ |I| &= \frac{2\, I_1}{Z_L + Z_o}\sqrt{\left(Z_L \sin\beta x\right)^2 + \left(Z_o \cos\beta x\right)^2} \end{aligned} \tag{36}$$

Recalling that we are dealing with cases where Z_L is real. Figure 5 shows the magnitudes of the voltage and current steady state waves with different load impedances. These are plotted as a function of x, the distance from the load. Except in the case Z_L=Z_o, there are minima and maxima in both V and I and also maxima of I occur where the minima of V are found, and vice versa.

For the cases $Z_L = \infty$ and $Z_L = 0$ (when $Z_L = \infty$, $\Gamma_L = +1$ and when $Z_L = 0$, $\Gamma_L = -1$), the magnitude of the reflection coefficient is 1 and as a consequence, the maximum possible amplitude of the steady state voltage for these two cases will be $2V_1$. If you compare the sign of the reflection coefficient for each case you should be able to distinguish why, at x = 0, one case is at its minimum value and the other is at its maximum value.

As the load impedance approaches the impedance of the line, the amplitude of the steady state voltage approaches V_1. When $Z_L = Z_o$ the steady state amplitude is V_1, since the reflection coefficient is 0.

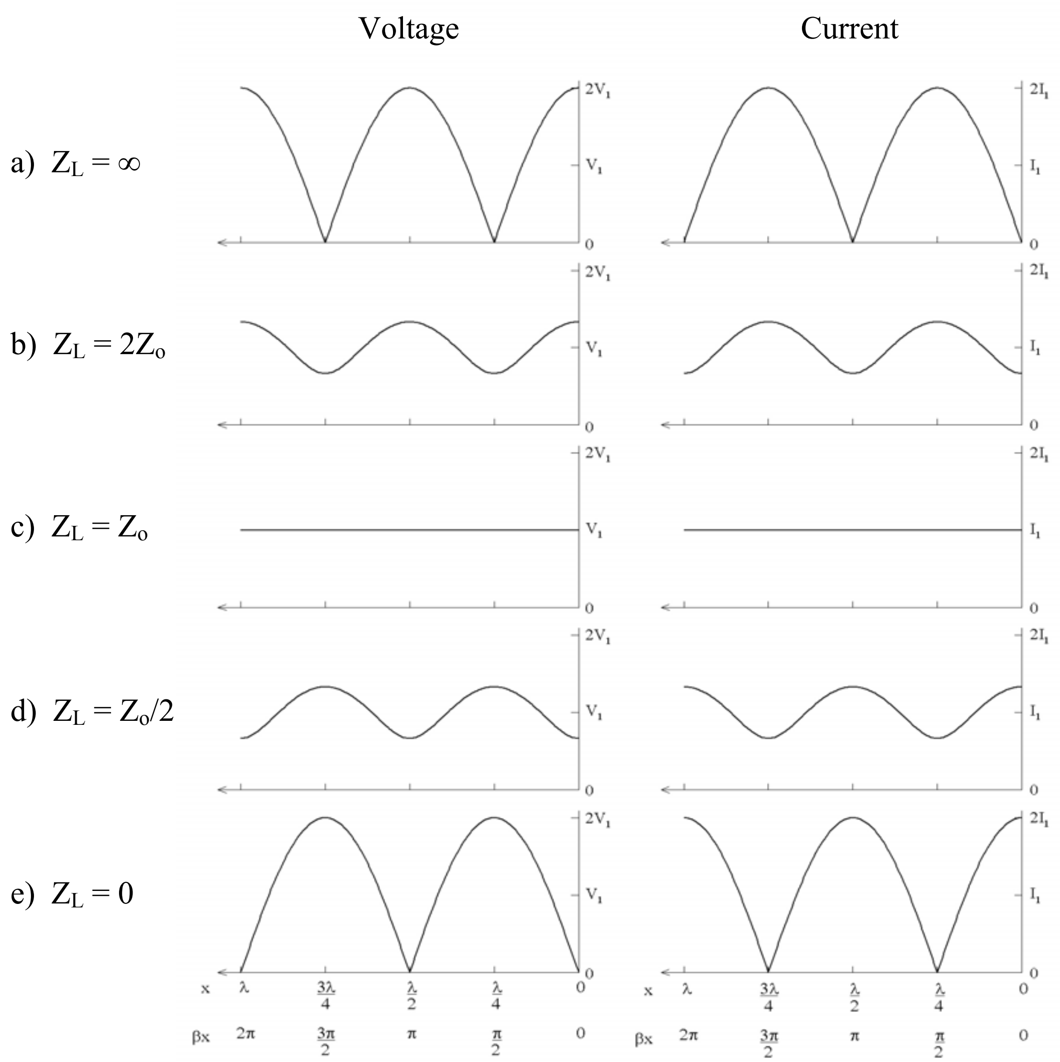


Figure 4. The magnitudes of the steady state voltage and current when various load impedances terminate the line.

The Voltage Standing Wave Ratio

We can develop our analysis further by determining the voltage standing wave ratio (VSWR). Expressed in terms of the maximum and minimum amplitude,

$$\text{VSWR} \equiv \frac{|V|_{max}}{|V|_{min}} \quad . \tag{37}$$

When $Z_L = Z_o$ there are no maxima or minima in the steady state voltage and VSWR=1. Since the steady state standing waves are produced by reflection, when $Z_L = Z_o$ the reflection coefficient is zero. Thus there is no reflection at the load.

When $Z_L \neq Z_o$ there is reflection at the load and standing waves are produced in the line as depicted by Figure 4. Referring to Figure 4, when $Z_L > Z_o$, a minima always occurs at $\beta x = \pi/2$ and maxima at $\beta x = 0$. Consequently,

$$\text{VSWR} = \frac{|V|_{\beta x=0}}{|V|_{\beta x=\pi/2}} = \frac{Z_L}{Z_o} \qquad \text{when } Z_L > Z_o \quad . \tag{38}$$

When $Z_L < Z_o$, a mimima always occurs at $\beta x = 0$ and maximum at $\beta x = \pi/2$. Consequently,

$$\text{VSWR} = \frac{|V|_{\beta x=\pi/2}}{|V|_{\beta x=0}} = \frac{Z_o}{Z_L} \qquad \text{when } Z_L < Z_o \quad . \tag{39}$$

In the cases represented by Equations 38 and 39, VSWR is greater than one. If we know Z_o (there are a limited number of commercially available coaxial transmission lines) and can determine whether $Z_L > Z_o$ or $Z_L < Z_o$ (recall we are dealing with Z_L real), then by measuring VSWR the load impedance can be found.

Given Equations 32 and 33, then for $Z_L > Z_o$ and $Z_L < Z_o$, VSWR can be written

$$\text{VSWR} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} \quad . \tag{40}$$

Solving for the magnitude of the reflection coefficient,

$$|\Gamma_L| = \frac{\text{VSWR}-1}{\text{VSWR}+1} \quad . \tag{41}$$

Consequently, a measurement of VSWR will not only yield Z_L if Z_o is known, it will also yield a value for the magnitude of the reflection coefficient.

Procedure

Transient waves: In this part of the experiment, you will observe the propagation of short impulses along a transmission line.

- Using the BNC tee, connect the pulse generator and channel 1 of the oscilloscope to the **L** end of the transmission line. The **0** end of the line should be left open-circuit.
- Observe the transmitted pulse along with any reflections on the oscilloscope. Print the waveform using the print function of your oscilloscope.
- Measure the time delay between the original pulse and the first reflection, and record this time. Make sure that the pulse width is smaller than the delay time (using the smallest pulse width on the generator).
- Attach the short-circuit termination to the **0** end of the transmission line, and repeat the above procedure.
- Repeat the measurements using the 4.7 Ω, 47 Ω, and 470 Ω terminators.

Standing waves: In this part of the experiment, you will observe the standing wave patterns produced by interference between transmitted and reflected sine waves.

- Remove the pulse generator and replace it with the sine-wave oscillator. Attach channel 2 of the oscilloscope to the **L/2** point on the line.
- With no load resistance attached to the **0** end of the line, vary the sine wave frequency and locate the lowest frequency null at the **L/2** point. Record this frequency.
- Increase the frequency and locate at least two more nulls at higher frequencies.
- Return to the first null frequency and measure the voltages (using the oscilloscope) at all five tap-points on the transmission line.
- Attach the short-circuit termination to the **0** end of the line and repeat the voltage measurements at the five tap points.
- Attach each of the other terminations 4.7 Ω, 47 Ω, and 470 Ω in turn, and repeat the voltage measurements at the five tap points.

Questions To Be Incorporated Into The Lab Report

Transient Waves: The transmission line is 81.5 m long. How long does it take an impulse to travel two lengths of the line? What is the propagation velocity of an electromagnetic wave in the line? What is the propagation factor ([propagation velocity]/[speed of light in a vacuum])?

Transient Waves: Measure the "a" and "b" diameters of the RG-58/U cable used in the transmission line (assume that *b* is equal to the outside diameter of the inner insulation). The impedance of RG-58/U cable is 53.5 Ω. What is the dielectric constant of the insulation?

Transient Waves: According to electromagnetic theory, the square root of the dielectric constant should be equal to the inverse of the propagation factor. How close is it?

Transient Waves: Discuss and explain the impulse waveforms you observed as you changed the termination resistance. Why does the waveform change its sign when $R<Z_o$?

Standing waves: In the standing wave portion of the experiment, what are the wavelengths of the waves at the three null frequencies?

Standing waves: Sketch the probable current and voltage waveforms for each value of termination resistance. Calculate the *VSWR* for each case.