SU(N) GAUGE THEORY AS STRINGS
ON DISCRETE RIEMANN SURFACES

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ABSTRACT

We discuss area preserving diffeomorphisms of a discrete torus with \( N^2 \) cells and the relation between such diffeomorphisms and SU(N) at finite \( N \). In particular certain discrete area preserving transformations that keep the lattice invariant and which form the group SL(2;\( Z_N \)) are clearly seen to correspond to a discrete subgroup of SU(N). The general SU(N) transformation is the isometry group of a new discrete Moyal bracket that replaces the Poisson bracket on the discrete surface. Using this observation we rewrite SU(N) gauge theory as the gauge theory of symplectic diffeomorphisms of the discretized surface. The gauge potential then has the labels of a string \( X(¿; ¾) \) with \( ¿; ¾ \) restricted to \( N^2 \) discrete points on the surface. Translation invariance on the torus reduces the number of gauge potentials down to \( N^2 - 1 \). The reduced QCD action for finite \( N \) can then be interpreted as a cut-off version of a string theory in the gauge det(\( °ij \)) = 1, where \( °ij \) is the induced metric on the surface. Sending \( N \) to infinity formally relates SU(1) QCD to a version of string theory. The present paper provides a formalism in which the suggested relation between these theories as well as between SU(1) and area preserving diffeomorphisms can be studied at the quantum level carefully as \( N \to 1 \).

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1{ Introduction

It has been observed that there is a relation between SU(N) as N → 1 and area preserving diffeomorphisms [1-5]. This relation is not true on arbitrary Hilbert spaces; it has been suggested that the relation may hold only for matrix elements of the generators when applied on a suitable set of functions [6]. This has to be understood in a sense analogous to the relationship between quantum mechanics and classical mechanics as \( \hbar \to 0 \). That is, unless we restrict ourselves to a suitable Hilbert space with the appropriate behaviour of the wavefunctions the relation is not true. In fact, the relation at the Lie algebra level breaks down for the high frequency modes of area preserving diffeomorphisms. The behaviour of the wavefunctions on which it is applied must damp the high frequency modes.

Based on the relation between SU(1) and area preserving diffeomorphisms, we suggested that large N reduced QCD can be cast into a form of string theory provided the gauge potential has "good" behaviour. In order to clarify this notion we have initiated a finite N approach which should lead to a careful understanding of the precise relation between SU(1) and area preserving diffeomorphisms, including the nature of the Hilbert space. The states of the Hilbert space of interest are in one-to-one correspondence with the gauge potentials. Thus, we want to determine the acceptable behaviour of the gauge potential and learn to what extent it represents a string. We would hope that the string behaviour coincides with the non-perturbative regime of the gauge theory.

In order to study the large-N behaviour carefully we will study latticized surfaces. The general surface of genus g will be discussed elsewhere while in this paper we will concentrate on the torus. In section 2 we introduce a latticized torus in "momentum" space and its dual lattice in "position" space. By "position" space we mean the \( \hat{\rho}; \hat{\lambda} \)space that parametrises the surface. The vertices of the "position" dual lattice correspond to the centers of the cells of the "momentum" lattice. These tori have N points or cells around each cycle, thus they contain a total of \( N^2 \) points or cells. This structure enables us to discuss the relation between SU(N) and the area preserving diffeomorphisms of the discretized torus at finite N. The dual "position" lattice is the relevant space for seeing the relation to string theory most directly. We nd that, at finite N, instead of the Poisson bracket which expresses the action of area preserving diffeomorphisms in the continuum case, a new discrete version of the Moyal bracket [7] expresses the action of SU(N) identically. Furthermore, a subgroup of area preserving diffeomorphisms that generates SL(2; \( \mathbb{Z}_N \)) is seen to be exactly equivalent to a discrete subgroup of U(N).

Armed with this finite N formalism, in section 3 we rewrite the SU(N) gauge theory
as a string-like theory defined on the latticized torus. We note some surprising symmetries of this action in section 4.

2 \{ SU(N) and diffeomorphisms on a discrete torus lattice

The continuum torus is described by two angles, $\frac{\pi}{i}$; $i = 1; 2$, with ranges $0 \leq \frac{\pi}{i} \leq \frac{\pi}{2}$. In string theory these are the usual $\varphi; \chi$ variables that parametrize the worldsheet. We concentrate on the torus because of the simplicity of the formalism in this case. A similar discussion on surfaces of arbitrary genus $g$ may be attempted. A function on the torus $f(\chi, \varphi)$ is periodic and therefore has an expansion in terms of the complete set of functions $\exp(i n_1 \varphi + i n_2 \chi)$, where $n_1; n_2 = 1; 2$, are in the set of all integers $\mathbb{Z}$. We define the discrete torus by limiting $n_1; n_2$ to a fundamental range, $n_1 = 0; 1; 2; \ldots; N_1 - 1$, and requiring a periodicity $n_1 \equiv n_1 + k_1 N_1$, where $k_1 \in \mathbb{Z}$ and $N_1$ is a prime number. That is we demand $n_1$ to belong to the set of integers modulo $N_1$, which is denoted by $\mathbb{Z}_{N_1}$. For most of our discussion we would be content more generally with any $N_1 = \text{odd integer}$, but as we shall see later a prime $N_1$ may be more interesting. This defines the "momentum" lattice. We now define the "position" lattice on the torus by writing $\chi = \frac{2\pi}{N_1} r_1$, with $r_1$ limited to the dual lattice with a fundamental domain $r_1 = 0; \frac{1}{2}; \frac{1}{2} + \frac{1}{2}; \ldots; N_1 - \frac{1}{2}$. That is $r_1$ belongs to $\mathbb{Z}_{N_1} + \frac{1}{2}$. Then $\frac{\pi}{i}$ belongs to the set $\frac{2\pi}{N_1}(\mathbb{Z}_{N_1} + \frac{1}{2})$ which we will denote by $S_{N_1}$, meaning the discretized circle. The importance of defining $\chi$ on the dual lattice will be understood after eq.(2.8) below. Each one of these lattices have $N_1^2$ points or cells, and they are dual to each other in the usual sense that the vertices of the dual lattice correspond to the center of the cells of the lattice.

With these definitions we see that the $N_1^2$ functions $f_n(\chi, \varphi) = \frac{1}{N_1} \exp(i n_1 \varphi + i n_2 \chi)$ have the required periodicities in both $\varphi; \chi$

$$f_{n+kN_1}(\chi, \varphi) = (i 1)^{k_1 + k_2} f_n(\chi, \varphi) \quad (2.1)$$

for any set of integers $k; n_1; n_2$ (the phase, which is independent of $n_1, \chi$ will drop out in physical quantities). They form a complete orthonormal basis on these lattices, i.e.

$$\sum_{n_1; n_2 = 0}^{N_1 - 1} f_n(\chi, \varphi) f^*_{n_1; n_2}(\chi, \varphi) = \delta_{n_1 \chi n_2; \varphi} ; \quad \sum_{r_1; r_2 = 0}^{N_1 - \frac{1}{2}} f_n(\chi, \varphi) f^*_{r_1; r_2}(\chi, \varphi) = \delta_{n_1 \chi n_2; \varphi} \quad (2.2)$$

For large $N_1$ we recover the continuum torus, and the discrete Kronecker delta function
tends to the continuum Dirac delta function

\[
\lim_{N \to \infty} \frac{h^2}{2\sqrt{N}} \sum_{\mathcal{q},0} i = \varepsilon(\mathcal{q},0) \lim_{N \to \infty} \frac{2\sqrt{N}}{N} \left[ \delta^{\frac{N}{2}} \right] : \quad (2:3)
\]

Any function on the discrete torus, such as the string coordinate \( X^1(\mathcal{q}) \), can be expanded in this complete set, thus

\[
X^1(\mathcal{q}) = \sum_{n} f_n(\mathcal{q}) ; \quad X^1_n = \sum_{\mathcal{q}} \frac{1}{2\sqrt{N}} \delta(\mathcal{q},n) : \quad (2:4)
\]

We now turn to a construction of \( N \times N \) matrices that form a basis for SU(\( N \)) [2]. Consider the \( N \times N \) Weyl matrices \( h \) and \( g \) that satisfy \( h^N = 1 = g^N \) and \( gh = hg \), where \( ! = \exp(i4\pi N) \) [8,9]. These matrices are explicitly given as \( h = \text{diag}(1; !; !^2; \cdots ; !^{N-1}) \) and \( g \) is the circular matrix, \( g^\mathcal{q} = \mathcal{q}_{\mathcal{q}+1} \) defined by identifying the indices \( i \) or \( j = N + 1 \) with \( 1 \), so that it has non-zero entries only above the diagonal and at the \( (i;j) = (N;1) \) location. The matrix \( g \) is diagonalized by a finite Fourier transform \( F^\mathcal{q} g F = h \), where \( F^\mathcal{q} = !^{\mathcal{q}} = N \). There are \( N^2 \) linearly independent powers of these matrices, \( h^{n_1} g^{n_2} ; n_1; n_2 = 0; 1; \cdots ; (N-1) \), that are unitary and close under multiplication. We take as a basis the normalized set

\[
(l^\mathcal{q})^\mathcal{q} = \frac{N}{4\sqrt{N}} (h^{n_1} g^{n_2})^\mathcal{q} : \quad (2:5)
\]

Note \((l^\mathcal{q})^\mathcal{q} = l_{\mathcal{q} / \mathcal{q}} = l_{N \cdot n_1; n_2} \cdot \). Using \( g^p h^q = h^q g^p \) we find the properties

\[
l_{\mathcal{q}} l_{\mathcal{r}} = N/4\sqrt{N} l_{\mathcal{q}+\mathcal{r}} ; \quad \text{Tr}(l_{\mathcal{q}}) = N^2/4\sqrt{N} \cdot \delta ; \quad \text{Tr}(l_{\mathcal{q}} l_{\mathcal{r}}) = N^3/4\sqrt{N^2 \cdot \delta} : \quad (2:6)
\]

Excluding the \( n_1 = 0 = n_2 \) identity matrix, the remaining ones are traceless and form a basis for SU(\( N \)) generators that close under commutation.

\[\text{If we consider a quantum mechanical model for a particle on a circle that is allowed to be at discrete positions} \quad Q = \frac{4\pi}{N}(0; 1; 2; \cdots ; N-1), \quad \text{then} \quad h = e^{iQ} \text{ and discrete translations are generated by} \quad g = e^{i4\pi N}. \quad \text{This provides a model for the matrices} \quad h; g \quad [3].\]
\[ [l_n; l_m] = i \frac{N}{2^{1/4}} \sin \left( \frac{2^{1/4}}{N} n \cdot m \right) l_{n+m}; \quad \text{(2.7)} \]

where \( n \cdot m = n_1 m_2 \). As claimed in ref.\[2\], as \( N \to \infty \) this algebra formally reduces to \( [l_n; l_m] = (n \cdot m) l_{n+m} \), which is recognized as the area preserving diffeomorphism algebra in the "momentum" basis. This claim breaks down for Fourier components for which \( n \cdot m = N \) is not small. Therefore, if we consider a general transformation by including the parameters \( \frac{n}{2^n} l_n \), then we are allowed to make the connection to area preserving diffeomorphisms only for a certain subspace of parameters that are well behaved at high frequencies. One of our purposes is to determine the nature of the large \( N \) parameter spaces in which the desired limit is correct. Since a gauge field has the same labels as \( \frac{n}{2^n} \), we hope to identify those spaces with the non-perturbative string-like behaviour of gauge fields.

For the purpose of connecting to area preserving diffeomorphisms and strings, we will find it more natural and interesting to define a representation of \( SU(N) \) in the dual "position" lattice. Thus, we take the finite Fourier transform

\[ (l(\xi))_i = \frac{N}{(2^{1/4})^2} \sum_{\nu, \xi = 2} \left( l_n \right)_i f_n(\xi) ; \quad (l_n)_i = \frac{(2^{1/4})^2}{N} \sum_{\nu, \xi = 2} (l(\xi))_i f_n(\xi) \quad \text{(2.8)} \]

It is here that we must notice that it is essential to define \( l(\xi) \) on the dual lattice \( S_1^N = \frac{2^{1/4}}{N} (Z_N + \frac{1}{2}) \). If it were not for the shift by \( \frac{1}{2} \), our \( l(\xi) \) would vanish, as seen by performing sums such as \( \sum_{n_1 \in S_1^N} e^{i 2 \pi n_1 \cdot r_1} N \) (fixed \( n_2 \)) and using \( h^N = 1 \).

The products and commutators of the matrices are now expressed in the dual lattice basis

\[ l(\xi) = \frac{1}{(2^{1/4})^2} \sum_{\nu, \xi = 2} \left( l(\xi) \right)_i f_n(\xi) \quad \text{(2.9)} \]
where the last line is demonstrated by using eq.(2.2). Taking the form of the delta function in eq.(2.3) into account, the last line justifies the normalization chosen in eq.(2.8). The derivatives are, of course, formal, since the functions are defined only on a discrete set of points, and their true meaning is defined via the finite Fourier transform. It is worth noting that the argument of the sine or exponential function in eq.(2.9) is proportional to the area formed by the three points \( \frac{\theta}{N} \frac{\theta}{N} \frac{\theta}{N} \) on the "position" torus. It is also noteworthy that these functions oscillate very fast as \( N \to 1 \) causing cancellations in the sums unless the area is of order \( \frac{1}{N} \), requiring the three points to approach each other and thus produce a local algebra on the torus. If we now take the formal large \( N \) limit, using eq.(2.3) and the last form of (9), we find, with \( i (\frac{\theta}{N}) \)

\[
\left[ L(\frac{\theta}{N}); L(\frac{\theta}{N}) \right] = i^{2ij} @ \frac{\theta}{N} \frac{\theta}{N} \frac{\theta}{N} @ L(\frac{\theta}{N}); \tag{2:10}
\]

which is the local form of the area preserving diffeomorphism algebra [6,10] and it can be rewritten as a Poisson bracket in the \( (\frac{\theta}{N}; \frac{\theta}{N}) = (\frac{\theta}{N}; \frac{\theta}{N}) \) variables

\[
\left[ L(\frac{\theta}{N}); L(\frac{\theta}{N}) \right] = i f \frac{\theta}{N} \frac{\theta}{N} \frac{\theta}{N} ; L(\frac{\theta}{N})g; \tag{2:11}
\]

Although we have now gained a different view of how the \( N \to 1 \) limit behaves, we must still find the parameter space \( \frac{2(\frac{\theta}{N})}{2(\frac{\theta}{N})} \left( \frac{2(\frac{\theta}{N})}{2(\frac{\theta}{N})} \right) \), that appears in the general SU(N) transformation \( R_{\frac{\theta}{N}} \frac{2(\frac{\theta}{N})}{2(\frac{\theta}{N})} \), for which this limit is valid.

The structure constants that we derived in eq.(2.9) have a close relation to the structure constants of the Moyal bracket [7], except that in our case we obtained a discrete version of it. The new result is that this discrete form of Moyal bracket is identically equivalent to SU(N). It is also clear that these structure constants satisfy the Jacobi identities since they come directly from matrix commutation rules.

The discrete Moyal bracket generates SU(N) and, for a discrete surface such as ours, it is the close relative of the Poisson bracket that generates area preserving diffeomorphisms on the continuous surface. Let us define the discrete Moyal bracket symbol \( f \frac{\theta}{N} \frac{\theta}{N} \frac{\theta}{N} \) applied on functions of the discrete "position" torus
\[ ff; g g^x(\mathcal{A}) = \frac{1}{N^2} \times \frac{N}{2\pi} \sin \left( \frac{2\pi}{N} \right) \mathcal{A} \]

where \( \mathcal{A} \) means that the derivatives are to be applied to the functions \( f, g \) respectively. This bracket satisfies Jacobi identities, as expected, since it is equivalent to SU(N) matrix commutation rules. The discrete \( N^2 \) points \( \frac{2\pi}{N} \) serve as the labels of the adjoint representation of U(N). As we shall later see, because of translation invariances there will be only \( N^2 - 1 \) independent parameters or gauge potentials.

Let the infinitesimal transformation of the function \( g(\mathcal{A}) \), which is in the adjoint representation, be defined as

\[ \pm g(\mathcal{A}) = f^2; g g^x(\mathcal{A}) \]  

where \( \mathcal{A} \) means that the derivatives are to be applied to the functions \( f, g \) respectively. This bracket satisfies Jacobi identities, as expected, since it is equivalent to SU(N) matrix commutation rules. The discrete \( N^2 \) points \( \frac{2\pi}{N} \) serve as the labels of the adjoint representation of U(N). As we shall later see, because of translation invariances there will be only \( N^2 - 1 \) independent parameters or gauge potentials.

Applying two infinitesimal transformations, and using Jacobi identities, we see that the composition law of group parameters is given by the discrete Moyal bracket \( f^2; g g^x(\mathcal{A}) \). This generates precisely infinitesimal transformations of SU(N) that are closely related to area preserving diffeomorphisms. The continuous \( N^2 - 1 \) parameters of SU(N) are identified as \( \mathcal{A} \) at the discrete points, or equivalently the coefficients of expansion of \( \mathcal{A} \) in the basis of eq.(2.1). Indeed by applying the discrete Moyal bracket to this basis (i.e. \( f_n(\mathcal{A}) = e^{i \mathcal{A} \cdot \mathcal{A}} N \)) we generate the SU(N) structure constants of eq.(2.7)

\[ f e^{i n \mathcal{A}} e^{i m \mathcal{A}} g^{x}(\mathcal{A}) = i \left( \frac{2\pi}{N} \right) \mathcal{A} \left( \frac{2\pi}{N} \right) \mathcal{A} \]  

We recall that area preserving diffeomorphisms on functions of continuous surfaces are represented just as in eqs.(2.12-13) except that the Poisson bracket replaces the discrete Moyal bracket.

Eq.(2.13b) proves that under the infinitesimal transformation (2.13a) our basis functions \( f_n \) transform like the adjoint representation of SU(N). We can now consider the \( N \) SU(N) transformation in the adjoint representation.
The new basis $f_0^0(\mathcal{A})$ satisfies identical periodicity properties to the original basis. This follows from (2.1) and the periodicity of the adjoint matrix $U_{n+kN,m} = U_{n,m+kN}$ that is due to periodicity on the torus$^\gamma$. Furthermore, as in any Lie algebra, these adjoint SU(N) transformations leave the structure constants (2.13b) invariant and therefore they are isometries of the discrete Moyal bracket (2.12).

The Moyal bracket was invented in the process of formulating quantum mechanics in phase space $(q;p)$ instead of the usual formulation in either position $(q)$ or momentum $(p)$ space. It was understood that the usual results of quantum mechanics for any value of $\hbar$ could be reproduced by applying the Moyal bracket to only a subset of functions of phase space [7,11]. The usual classical limit of quantum mechanics that follows in the limit $\hbar \to 0$ is valid within this subspace and then the Moyal bracket reduces to the Poisson bracket of classical mechanics in this space. In our case the role of phase space $(q;p)$ is played by $\mathcal{A}$ and $\hbar$ is replaced by $\frac{4\pi}{N}$. Therefore, it may be useful to pursue this analogy to find an appropriate space for the SU(N) parameters of transformation $^\gamma(\mathcal{A})$. The \"good\" parameter subspace is expected to be analogous to the \"good\" wavefunctions of the Moyal bracket formulation of quantum mechanics. In such a subspace we are allowed to substitute SU(N) by area preserving diffeomorphisms as $N \to 1$.

We discuss one last mathematical observation in this section. There is a discrete subgroup of diffeomorphisms that transforms the points or cells on the torus. This is the group $\text{SL}(2;\mathbb{Z}_N)$ represented by $2 \times 2$ matrices of integers modulo $N$. The transformation is

$$\begin{pmatrix} \mu & \| \mu \\ \nu & \| \nu \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mu & \| \mu \\ \nu & \| \nu \end{pmatrix} \equiv \begin{pmatrix} a \nu+b \mu \\ c \nu+d \mu \end{pmatrix} \mod(S_N^1)$$

These transformations are automorphisms of the SU(N) algebra since the cross products appearing in the structure constants of eqs.(2.7,9,12,14) remain invariant. The cross products represent areas on the torus, so these transformations are indeed area preserving.

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$^\gamma$ A unitary SU(N) matrix in the fundamental representation may be written as $U^i_n = \sum_p u^i_n(l^i_n) \delta_{ij}$ with the conditions $\pm \nu = \sum_m e^{\pm 2\pi \nu N} U_{n+m} u^i_m$ that follow from unitarity. The corresponding matrix in the adjoint representation is now obtained as $U_{nm} \to \text{Tr}(l^i_n U^i_l \nu \nu) \equiv \sum_p e^{(m\nu+n\nu+p\nu+p)} U_{mn} u^i_{p+n+m}$. 

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di@eomorphisms of the discrete surface. One can find an $N \times N$ matrix representation of this group such that

$$R(A)l(\frac{\lambda}{2})R_1(A) = l(A\frac{\lambda}{2}) ; \quad R(A)l_nR_1(A) = l_{A,n}$$

(2.16)

where $A\frac{\lambda}{2}$ is the action of the $2 \times 2$ matrix $A$ given in eq.(2.15), and $R(A)$ is a discrete matrix in $U(N)$ that represents it [3]. It is then clear that this discrete subgroup is an automorphisms as seen by applying similarity transformations in eqs.(2.7,2.9). It is satisfying to see that the explicit di@eomorphism of eq.(2.15) belongs to the larger continuous parameter group SU(N). Therefore, there is this particular discrete subgroup of SU(N), represented by $R(A)$, that is rmly related to area preserving di@eomorphisms of the discrete torus even before we take the large $N$ limit.

The generators of SU(N) are classi_ed by this symmetry group. In particular we may ask: under what circumstances do all the SU(N) generators form a single irreducible representation under the subgroup SL(2; $Z_N$) ? That is, when is it possible to obtain all the generators of SU(N), $l_n$ or $l(\frac{\lambda}{2})$ by applying this automorphism group to a single generator, say $l_{1;0} \rightarrow h$

$$l_n \rightarrow h^{n_1}g^{n_2} \rightarrow R(A)hR_1(A) :$$

(2.17)

This amounts to the question of whether we can find matrices $A$ that satisfy

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mod N$$

(2.18)

such that $\det(A) = 1 \mod (N)$ when $n_1; n_2$ take every value in $Z_N$ (except $n_1 = 0 = n_2$). That is, with $a = n_1$; $c = n_2$ we need to find a solution to the determinant equation $n_1d = n_2b = 1 + kN$, with $k$ any integer and $d;b 2 Z_N$. In particular when we specialize to $n_2 = 0$ (or $n_1 = 0$) we nd that there is a solution only and only if $N = \text{prime}$. If there is a solution to these special cases then there is always a solution parametrized by three integers $n_1;n_2;n_3 2 Z_N$ in the form

$$A = \begin{pmatrix} n_1 & n_1n_3 & m_2 \pm n_1;0 \\ n_2 & n_2n_3 + m_1 \mod N \end{pmatrix}$$

(2.19)
where $m_1; m_2$ are fixed by requiring $n_1 m_1 = 1 + k_1 N$; $n_2 m_2 = 1 + k_2 N$ for any integers $k_1; k_2$. So, we claim that

$$\det(A) = 1 \mod(N) \quad \text{for all values of } n_1, n_2 \in \mathbb{Z}_N \quad \text{iff} \quad N = \text{prime}; \quad (2:20)$$

There may exist a well known theorem that proves this fact. We have verified this proposition by computer up to integers $N$ of order 2,000. For non-prime values of $N$ we cannot recover all the generators by $\text{SL}(2; \mathbb{Z}_N)$ transformations by starting from a single generator. This means that the $\text{SU}(N)$ generators are classified in more than one irreducible representation of $\text{SL}(2; \mathbb{Z}_N)$ if $N$ is not prime. Thus, for $N = \text{prime}$ we have the interesting situation of obtaining all $\text{SU}(N)$ generators (but not arbitrary linear combinations) by starting from a single infinitely small $\text{SU}(N)$ transformation and then applying area preserving diffeomorphisms on the discrete torus. This should be helpful in the analysis of the relationship between the continuous transformations of $\text{SU}(N)$ and the continuum area preserving transformations of the torus as $N \to 1$.

3{SU($N$) gauge theory with discrete Moyal brackets.

In the previous section we labelled the adjoint representation of SU($N$) in terms of the $N^2$ points on the discrete $\mathbb{Z}_N$ torus. We would like to rewrite the SU($N$) gauge theory in this basis for the purpose of making a connection to string theory. The gauge potential may be written as a matrix $(A_i (x^j)) ; i; j = 1; 2; \ldots; N$ or as a function on the discrete surface $A_i (\mathcal{M}_x)$. They are connected by

$$ (A_i)_i = \left. X \right|_{\mathcal{M}_x} \quad A_i (\mathcal{M}_x) \quad ; \quad A_i (\mathcal{M}_x) = \left. \frac{4(2^\frac{1}{4})^6}{N^5} \text{Tr} l(\mathcal{M}_x) \right|_{\mathcal{M}_x} \quad (3:1)$$

where $\left. X \right|_{\mathcal{M}_x}$ is a normalization to be determined. The field strength becomes

$$ (F_{i\theta})_{ij} = \left. @ (A_{\theta})_{ij} \right|_{\mathcal{M}_x} \quad i; j = 1; 2; \ldots; N \quad \left. @ (A_{\theta})_{ij} \right|_{\mathcal{M}_x} = \left. X \right|_{\mathcal{M}_x} \quad F_{i\theta} (\mathcal{M}_x) \quad ; \quad (3:2)$$

$$ F_{i\theta} (\mathcal{M}_x) = \left. @ (A_{\theta})_{ij} \right|_{\mathcal{M}_x} \quad @ (A_{\theta})_{ij} = \left. \frac{N^2}{(2^\frac{1}{4})^2} f A_{ij} ; A_{\theta} g^a \right|_{\mathcal{M}_x}$$
where the discrete Moyal bracket of eq.(2.12) appears. The normalized action takes the form

$$ S = i \frac{N}{4g^2} \int d^3x Tr \, F_{10}^2 = i \frac{N^{6-2}}{16(2\sqrt{g}^6g^2) \int d^3x \, F_{10}(\frac{\sqrt{g}}{2})^2 :}$$

We would like to study the reduced version of this theory following the ideas of [6]. The reduced model for any large N theory was invented in 1982 [12]. It reproduces the planar graphs of the original theory and gives the space-time dependence of Green's functions by performing the path integral at a single space-time point. We think that the reduced model, with some modification \(^f\), is capable of reproducing the momentum dependence of non-planar diagrams as well. In the reduced model space-time derivatives of matrix fields are replaced by a commutator with a fixed (quenched) momentum matrix \(P_{10}\).

In the gauge theory the covariant derivative becomes

$$ a_{\vec{v}} = P_{10} + A_{\vec{v}} \cdot \vec{D}_{10},$$

where \((A_{\vec{v}})_{ij}\) is the matrix that represents the gauge field at \(x^1 = 0\). However, since the gauge field always appears in a covariant derivative, only the matrix \(a_{\vec{v}}\) appears in all expressions. This leads to the Yang-Mills field strength

$$ (F_{10}) = [\vec{D}_{10}; \vec{D}_{10}] ! [a_{\vec{v}}, a_{\vec{w}}]_{ij} \quad (f_{10})_{ij};$$

and the reduced gauge theory action

$$ S_{red} = i \frac{1}{4} \int d^3x \frac{N}{\sqrt{\delta^2}} \text{Tr}(f_{10} f_{10}^\dagger);$$

where \(\delta\) is a cut-off. There is a constraint between the eigenvalues of the matrix \(a_{\vec{v}}\) and those of \(P_{10}\), \(a_{\vec{v}} = U_{\vec{v}} P_{10} U_{\vec{v}}^{\dagger}\), where \(U_{\vec{v}}\) diagonalizes \(a_{\vec{v}}\) separately for each \(\vec{v}\). The \(U_{\vec{v}}\) grows because of the quenching of momenta. This is due to the fact that index lines can wind around \(2g\) independent cycles. However, for dense diagrams with order \(N^g\) propagators, the mismatch (contributed by the missing \(2g\) momenta) occurs in order \(N^g\) propagators. Ideally one should modify the reduced theory to correct for the mismatch but here I will proceed hoping that as \(N \downarrow 1\) this mismatch can be ignored. This question will be analysed in more detail elsewhere.
are reduced Wilson lines. This gives rise to a factor in the integration measure $f(a) = R Q \prod U_{i} \pm(a_{i} U_{i} P_{i} U_{i}^{\dagger})$, and yields the reduced path integral

$$Z \sim \int D a \ f(a) \exp(iS_{\text{red}}(a)) : \tag{3.6}$$

We now rewrite the reduced theory in terms of the discrete torus. We expand

$$f(a) = R Q \prod U_{i} \pm(a_{i} U_{i} P_{i} U_{i}^{\dagger}) \rightarrow X_{i}(\phi(\bar{\phi}))^{a_{i}} \quad ; \quad X_{i}(\phi(\bar{\phi}))^{a_{i}} = \frac{4(2\sqrt{2})^{7}}{\pi^{2} N^{5}} \ln(\phi(\bar{\phi}))^{a_{i}} \quad : \quad \tag{3.7}$$

where we will interpret $X_{i}(\phi(\bar{\phi}))$ as a (dimensionless) string field on the discrete torus. The reduced field strength can be written in terms of the discrete Moyal bracket as

$$(f_{10})^{a_{i} b_{i}} = \frac{\pi^{2} N^{2-2} N}{(2\sqrt{2})^{4}} \ln(\phi(\bar{\phi}))^{a_{i}} \quad ; \quad X_{i}(\phi(\bar{\phi}))^{a_{i}} = \frac{4(2\sqrt{2})^{7}}{\pi^{2} N^{5}} \ln(\phi(\bar{\phi}))^{a_{i}} \quad : \quad \tag{3.8}$$

We interpret $f X_{i} ; X_{0} g$ as the string area density defined on the discrete torus. The formal large $N$ limit of this quantity is indeed the area density expressed as a Poisson bracket $f X_{i} ; X_{0} g = \partial X_{i} \ partial X_{0} ; X_{i} \ partial X_{0}$. Finally the reduced action takes the form

$$S_{\text{red}} = \int \frac{(2\sqrt{2})^{4}}{16 g^{2}(\pi)} \frac{4^{3}}{N^{5}} \ln(\phi(\bar{\phi}))^{a_{i}} \quad ; \quad \tag{3.9a}$$

This is to be compared to the string action

$$S \sim d^{23} \partial \text{det}^{(0)} \quad \Rightarrow \quad d^{23} f X_{i} ; X_{0} g^{2} ; \quad \tag{3.9b}$$

where semi-classically $\partial_{ij} = \partial X_{i} \ partial X_{j}$ is the induced metric. This action reproduces the Euclidean string equations of motion in the gauge $\text{det}^{\circ} = 1$. By this gauge choice one breaks the general diffeomorphism invariance of string theory down to the subgroup of area preserving transformations characterized by $\partial^{4} = \partial(\phi(\bar{\phi})^{a_{i}} ; \partial^{21} = 0$. The action (3.9b) is quartic in $X_{i}$ and needs to be regularized in order to treat it quantum mechanically.
Eq.(3.9a) can then be considered as the cut-off version of this string action which is now defined on the latticized torus. We have thus related a cut-off version of string theory to the reduced action of large-$N$ QCD. Both of these theories require that we take the limit $N \to \infty$ at the end of a quantum calculation. To perform quantum calculations one may still choose gauges for the remaining SU($N$) gauge invariance.

After some manipulations the action (3.9a) can be written as

$$\sum_{i} \frac{1}{2} S_{N} X_{i} X_{i} (X_{i} X_{i} + X_{i} \delta X_{i}) = \sum_{i} \frac{1}{2} \left( \sum_{i} \frac{1}{2} (\frac{2}{N})^{2} X_{i} X_{i} + \frac{1}{2} (\frac{2}{N})^{2} X_{i} \delta X_{i} \right) \cos \frac{2\pi}{N} \left( \frac{
}{2} \leq \frac{
}{2} \leq \frac{
}{2} \right) \pm \frac{\n}{2} \pm \frac{\n}{2} \pm \frac{\n}{2}$$

(3.10)

Except for the sign changes in the arguments of the delta function. If we write the action in the "momentum" torus by using the finite Fourier transform of eq.(2.4) we find remarkably the same form

$$\sum_{i} \frac{1}{2} S_{N} X_{i} X_{i} (X_{i} X_{i} + X_{i} \delta X_{i}) = \sum_{i} \frac{1}{2} \left( \sum_{i} \frac{1}{2} \left( \frac{2}{N} \right)^{2} X_{i} X_{i} + \frac{1}{2} \left( \frac{2}{N} \right)^{2} X_{i} \delta X_{i} \right) \cos \frac{2\pi}{N} \left( \frac{
}{2} \leq \frac{
}{2} \leq \frac{
}{2} \right) \pm \frac{\n}{2} \pm \frac{\n}{2} \pm \frac{\n}{2}$$

(3.11)

except for the sign changes in the arguments of the delta function. Note that the arguments of the cosine in eqs.(3.10, 3.11) is proportional to the area of the parallelogram formed by the four points $(\frac{\n}{2}; \frac{\n}{2}; \frac{\n}{2}; \frac{\n}{2})$ on the momentum lattice or $(\frac{\n}{2}; \frac{\n}{2}; \frac{\n}{2}; \frac{\n}{2})$ on the position lattice.

4{Symmetries of the action.

The action constructed above has a number of symmetries that are potentially useful in analysing its quantum properties. These include duality, global translation invariance on the torus as well as in target space, discrete diffeomorphisms and continuous SU($N$). These are described below.
1) As seen by the relationship between eqs. (3.10, 3.11), the action has essentially the same form in "momentum" or "position" space. That is there is a "duality" invariance under the transformation of (2.4).

2) The action is invariant under a global translation on the torus: \( X^0(\tilde{\varphi}) = X^1(\tilde{\varphi} + \mathbf{a}) \) where \( \mathbf{a} \) is any vector on the dual lattice. There is also a similar translation invariance in the momentum lattice where we may translate \( \mathbf{n}_2; \mathbf{n}_3 \) oppositely to \( \mathbf{n}_1; \mathbf{n}_4 \).

3) Global translation invariance in target space, \( X^0(\tilde{\varphi}) = X^1(\tilde{\varphi} + \mathbf{q}) \) is an invariance of the action. This is seen from (3.10) by noting that the sums vanish in the terms proportional to \( \mathbf{q} \). This allows the average position of the string to be set equal to zero and is equivalent to the tracelessness of the matrix \( a_{\bar{1}1} \). This is expected since the trace part of \( a_{\bar{1}1} \) drops out in the reduced action (3.4,3.5).

4) Discrete diffeomorphisms of the torus as in eq.(2.15) leave the action invariant under the transformation \( X^0(\tilde{\varphi}) = X^1(A\tilde{\varphi}) \), and similarly in momentum space. This is the discrete version of a subset of reparametrizations of the string. As seen in eq.(2.16) these form a discrete subgroup \( SL(2;\mathbb{Z}_N) \) of \( SU(N) \) and are indeed area preserving transformations. These transformations preserve the lattice.

5) Continuous \( SU(N) \) symmetry is an invariance of this action. As seen in section 2, this is related to the original \( SU(N) \) gauge symmetry and is expressed as in (2.13) \( \pm X^0(\tilde{\varphi}) = f^2; X^1 \tilde{\varphi} \), where the discrete Moyal bracket is given in eq.(2.12). In the large \( N \) limit, for well behaved \( \tilde{\varphi} \), this will reduce to the continuum area preserving reparametrizations expressed as a Poisson bracket.

5{Discussion.

We have formulated area preserving diffeomorphisms of a discrete torus with \( N^2 \) cells and discussed the relation between such diffeomorphisms and \( SU(N) \) at finite \( N \). \( SU(N) \) gauge theory is then reinterpreted as the gauge theory of area preserving diffeomorphisms of the discretized surface. If we then consider the string action in the gauge \( \text{det}(\tilde{\varphi}) = 1 \) and a cut is introduced by latticizing the torus, it is seen that the cut string action coincides with the reduced gauge theory action at finite \( N \).

Generalizations of these observations to surfaces of any genus \( g \) have been obtained. This is done by utilizing the explicit solutions for the wavefunctions of a particle moving on
a genus $g$ surface in the presence of a magnetic field (the quantum Hall effect Hamiltonian), which we have solved recently. The zeroes of these wavefunctions naturally define a lattice on the genus $g$ surface. For every genus one sees that there are many ways to take the limit $\text{SU}(N) \to \text{SU}(1)$. As this involves extensive discussion and involves formalism not introduced in the current paper, we will present these results in a separate publication.

The present paper provides a cut-off theory in which the $N \to 1$ limit can be studied carefully. The suggested relation between QCD and string theories as well as between $\text{SU}(1)$ and area preserving diffeomorphisms can now be better analyzed at the quantum level. Of particular interest is the nature of the string theory as the cut-off is removed $N \to 1$ and its relation to ordinary string theory in this limit. The double scaling limit of this matrix model is also of interest since it could provide non-perturbative insight into both string theory and QCD. Work along these lines will be discussed elsewhere.

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