Dear Organizers,

Last month, I sent you my contribution to the proceeding of the conference, String '95. Recently I noticed some minor errors in my manuscript. If it is still possible, I would like to make corrections. The following is a revised version of the manuscript. I would appreciate if you could replace it with the previous version. Thank you for your cooperations.

Sincerely, Hirosi Ooguri Department of Physics UC Berkeley

LOOP AMPLITUDES OF $N=2$ STRING

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Abstract

This report is based on my work in collaboration with Cumrun Vafa [1]. Using the $N=4$ topological reformulation of $N=2$ strings, we compute all loop partition function for special compactifications as a function of target moduli. We also reinterpret $N=4$ topological amplitudes in terms of slightly modified $N=2$ topological amplitudes.

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1. Introduction

One of the simplest types of string theories is $N = 2$ strings. It lives in four dimensions, and it has finite number of particles in the spectrum. Moreover it describes self-dual geometries and Yang-Mills fields [2] [3], which are conjectured to describe, through reduction, all 2 and 3 dimensional integrable models. Moreover the 4 dimensional $N = 2$ string itself seems to correspond to an integrable theory, as is evidenced by perturbative vanishing of scattering amplitudes beyond three point functions.

Given all these connections, it seems very important to understand $N = 2$ string amplitudes. In this paper we consider this question and find, rather surprisingly, that one can compute, at least in special cases, the all genus partition function of $N = 2$ strings. This seems to be another evidence for the quantum integrability of self-dual theories. More specifically we consider compactifications of $N = 2$ strings on $T^2 \times R^2$. Using the reformulation of $N = 2$ strings in terms of $N = 4$ topological strings[4], allows one to develop techniques to compute it.

For low genus, this can be done more or less directly, because the structure of the amplitudes are so simple. However for $g > 2$ the story gets more complicated. In such cases we have found a modified version of the harmonicity equation of [4] for which the boundary contributions cancel, and are strong enough to yield the genus g partition function up to an overall constant. Specialized to $g = 1, 2$ this result agrees with explicit computations of the amplitudes. This is somewhat analogous to the method used in [5] to compute the topological $N = 2$ string amplitudes, with the replacement of holomorphic anomaly with harmonicity equation.

Another aspect of $N = 2$ string, is the topological interpretation of what it is computing. We show that quite generally $N = 4$ topological strings, are a slightly (but crucially) modified form of $N = 2$ topological string amplitudes. This allows us to give a more clear interpretation of what topological quantities does the partition function compute. In particular we see quite explicitly in the cases of genus 1 and 2 in the example of $T^2 \times R^2$ what these topological quantities are, and moreover reproduce in yet another way, the partition function itself by direct topological evaluation.

2. $N = 2$ String and Harmonicity

In this section we briefly review aspects of $N = 2$ strings which are relevant for this paper. $N = 2$ strings was first studied in the early days of string theory [6] and its study was resumed with the surge of interest in string theory [7]. It was discovered relatively recently [2], [3] that $N = 2$ string theory has a rich geometric structure related to self-duality phenomena. In particular its critical dimension is four (2 complex dimensions), and the closed string theory describes self-dual gravity, whereas heterotic and open string versions describe self-dual gauge theories in four dimensions coupled to self-dual gravity.

Some of these aspects were further studied [8]. More recently it was shown [4] that the loop amplitude computations in $N=2$ theories can be simplified by proving their equivalence to a new topological string based on the small $N=4$ superconformal algebra. In this way the ghosts are eliminated and at the same time the matter fields are topologically twisted; this makes computations much easier. The main aim of this paper is to further elaborate on the meaning of the $N=2$ string amplitudes in light of this development. In this section we will give a brief review of the topological reformulation of [4] referring the interested readers for the detail to that paper. We will mainly concentrate on the closed string case. The generalization to other cases (heterotic and open) are straight forward.

$N=2$ strings are obtained by gauging the $N=2$ local supersymmetry on the world-sheet. This consists of the metric $g_{\mu\nu}$, two supersymmetric partners of spin 3/2, $\psi^{\pm}_{\mu\alpha}$ and one $U(1)$ gauge field A_μ. In the standard fashion, these give rise to a pair of fermionic ghost (b,c) of spin 2, two pairs of bosonic superghosts $(\beta^{\pm},\gamma^{\pm})$ of spin 3/2 and another pair of fermionic ghost $(\tilde{b},\tilde{c})$ of spin 1. The total ghost anomaly is $c=-6$, which is cancelled by a matter with $c=6$, corresponding to a superconformal theory in 4 dimensions. The vacua of $N=2$ strings consist of theories in 4d which have Ricci-flat metric [2]. These theories will necessarily have an extended symmetry, by including the spectral flow operators, to the small $N=4$ superconformal algebra with $c=6$ ($\hat{c}=2$).

The $N=4$ algebra consists of an energy momentum tensor T of spin 2, an $SU(2)$ current algebra of spin 1, whose generators are denoted by J^{++},J,J^{--} and 4 spin 3/2 supercurrents which form two doublets $(G^-,\tilde{G}^+)$ and $(\tilde{G}^-,G^+)$ under the $SU(2)$ currents. The supercurrents within a doublet have no singularities with each other, while the oppositely charged supercurrents of the different doublets have singular OPE (and in particular give the energy momentum tensor). Moreover G^+ and $\tilde{G}^+$ have a singular OPE with a total derivative as the residue:

$$G^+(z)\tilde{G}^+(0)\sim\frac{\partial J^{++}(0)}{z}$$

Note in addition that

$$\tilde{G}^+=G^-(J^{++}) \tag{2.1}$$

which follows from the fact that $(G^-,\tilde{G}^+)$ form an $SU(2)$ doublet. Also note that J^{++} is the left-moving spectral flow operator. This in particular implies that the chiral field V corresponding to the volume form of the superconformal theory can be written as

$$V=J^{++}_LJ^{++}_R \tag{2.2}$$

Together with (2.1) this means that

$$G^-_LG^-_RV(z,\bar{z})=\tilde{G}^+_L\tilde{G}^+_R(z,\bar{z}) \tag{2.3}$$

It is important to note that the choice of two doublets among the four supersymmetry currents is ambiguous: In particular there is a sphere worth of inequivalent choices given by

$$
\begin{aligned}
\widehat{\tilde{G}^{+}}(u) &= u^1 \tilde{G}^{+} + u^2 G^{+} \\
\widehat{G^{-}}(u) &= u^1 G^{-} - u^2 \tilde{G}^{-} \\
\widehat{\tilde{G}^{-}}(u) &= u^{2*} \tilde{G}^{-} - u^{1*} G^{-} \\
\widehat{G^{+}}(u) &= u^{2*} G^{+} + u^{1*} \tilde{G}^{+}
\end{aligned}
\tag{2.4}
$$

where

$$|u^1|^2 + |u^2|^2 = 1$$

and where the complex conjugate of u_a is $\epsilon^{ab} u_b^*$ (i.e. $\overline{(u^1)} = u^{2*}$ and $\overline{(u^2)} = -u^{1*}$ where $*^2 = -1$). Note that we could do this rotation for left and right $N = 4$ algebras independently, and we will use u_L, u_R to denote the left- and right-moving choices for the rotation.

A theory with $N = 4$ superconformal structure can be deformed, preserving the $N = 4$ structure using chiral field of (left,right) charge (1,1). There are four deformations that can be made out of a given chiral field ϕ^i:

$$S \to S + \int t_i^{11} G_L^- G_R^- \phi^i - t_i^{21} \tilde{G}_L^- G_R^- \phi^i - t_i^{12} G_L^- \tilde{G}_R^- \phi^i + t_i^{22} \tilde{G}_L^- \tilde{G}_R^- \phi_i$$

Note that for unitary $N = 4$ theories, these deformations are pairwise complex conjugate. In particular there exists a matrix M_i^{j*} so that

$$t_i^{ab} = \epsilon^{ac} \epsilon^{bd} M_i^{j*} t_j^{\bar{c}d}$$

with $MM^* = 1$.

The $N = 2$ string amplitudes are computed by integration of the string measure over the $N = 2$ supermoduli. The bosonic piece of this moduli consists of the moduli of genus g Riemann surfaces as well as the g-dimensional moduli of $U(1)$ bundles with a given instanton number n. For a fixed instanton number the dimension of $\beta^{\pm}$ zero modes gives the dimension of supermoduli. Since they are charged under the $U(1)$ this dimension will depend on the instanton number. In particular the dimension of these supermoduli is $(2g - 2 - n, 2g - 2 + n)$ for the (β^+, β^-) zero modes. In particular this means that $|n| \leq 2g - 2$ in order to get a non-zero measure. Even though geometrically not obvious, it turns out that we can also assign independent left-moving and right-moving instanton numbers. So at each genus g we have to compute the string amplitudes $F^g_{n_L, n_R}$ with

$-2g+2 \leq n_L, n_R \leq 2g-2$. It is convenient to collect these amplitudes in terms of a function on u-space. Let

$$
\begin{aligned}
&F^g(u_L,u_R) = \\
&= \sum_{-2g+2\leq n_L,n_R\leq 2g-2} \binom{4g-4}{2g-2+n_L}\binom{4g-4}{2g-2+n_R} \cdot F^g_{n_L,n_R} \times \\
&\quad \times (u^1_L)^{2g-2+n_L}(u^1_R)^{2g-2+n_R}(u^2_L)^{2g-2-n_L}(u^2_R)^{2g-2-n_R}
\end{aligned}
$$

The result of [4] is that F^g can be computed by

$$
\begin{aligned}
F^g(u_L,u_R) = \int_{\mathcal{M}_g} \langle [\prod_{A=1}^{3g-3}(\mu_A, \widehat{G^-_L}(u_L))(\bar{\mu}_A, \widehat{G^-_R}(u_R))] \int_\Sigma J_L J_R \times \\
\times \,[\int_\Sigma \widehat{\tilde{G}^+_L}(u_L)\widehat{\tilde{G}^+_R}(u_R)]^{g-1}\rangle
\end{aligned}
\tag{2.5}
$$

where Σ denotes the Riemann surface and $\mathcal{M}_g$ denotes the moduli of genus g surfaces and μ_A denote the Beltrami differentials. In this expression there are no ghosts left over and the $N=4$ matter field is topologically twisted, i.e. the spin of the fields are shifted by half their charge, so in particular $G^+, \tilde{G}^+$ have spin 1 and $G^-, \tilde{G}^-$ have spin 2 and J^{++} has spin 0 and J^{--} has spin 2.

Let us give a rough outline of how the above correspondence between $N=2$ string amplitudes and the $N=4$ topological amplitude, defined above, arises. The simplest case of constructing this measure corresponds to the $n_L = n_R = 2g-2$. In this case we have no β^+ zero modes, and $(4g-4)$ β^- zero modes. If we had instanton number $(g-1)$, it would have been equivalent to twisting the fields, by the definition of topological twisting (identifying gauge connection with half the spin connection). So for instanton number $(2g-2)$, we can view the amplitudes as being computed in the topologically twisted version but with an addition of $(g-1)$ instanton number changing operators inserted. Note that the matter part of the instanton number changing operator is J^{++}. In the topologically twisted measure, the (β^-,γ^-) ghost system have the same spin as (b,c) and the (β^+,γ^+) have the same spin as $(\tilde{b},\tilde{c})$, and since they are of the opposite statistics they cancel each other out as far as the non-zero modes are concerned. The zero modes can also be canceled out by a judicious choice of the position of picture changing operators. We have $(4g-4)$ picture changing operators inserted for integration over the supermoduli which are accompanied from the matter sector with G^-. $(3g-3)$ of them get folded with the Beltrami differentials in cancelling the zero modes of b. The integration over the $U(1)$ moduli is traded with integration over g operators on Riemann surfaces: $(g-1)$ of them come from operators where $(g-1)$ of the instanton changing operators have converted G^- into $\tilde{G}^+$ and the last one is simply the current J. This would give the correspondence at the highest instanton numbers and the rest are obtained by performing an $SU(2)$ rotation on the $N=2$ string side and seeing that it corresponds to changing the instanton numbers.

In the paper [4], it was pointed out that the $N=2$ string amplitude $F^g(u_L,u_R)$ would solve

$$\begin{aligned} \epsilon^{ab}\frac{\partial}{\partial u_L^a} D_{t^{bc}} F^g(u_L,u_R) &= 0 \\ \epsilon^{ab}\frac{\partial}{\partial u_R^a} D_{t^{cb}} F^g(u_L,u_R) &= 0 \end{aligned} \tag{2.6}$$

provided we could ignore contributions from the boundary of the moduli space $\mathcal{M}_g$ and contact terms in operator products. We have examined this assumption carefully [1], and it turned out that there are in fact contact terms which spoil (2.6). Fortunately there is a weaker version of the equation

$$\epsilon^{ab} u_R^c \frac{\partial}{\partial u_L^a} D_{t^{bc}} F^g(u_L,u_R) = 0 \tag{2.7}$$

$$\epsilon^{ab} u_L^c \frac{\partial}{\partial u_R^a} D_{t^{cb}} F^g(u_L,u_R) = 0. \tag{2.8}$$

in which these contact terms are canceled out. In the case when the target space is $T^2\times R^2$, these equations are strong enough to determine F^g completely up to a constant at each g independent of the moduli of T^2.

3. $N=2$ String Amplitudes on $T^2\times R^2$

At genus one, the $N=2$ string amplitude on $T^2\times R^2$ has been computed in our previous paper [2] as

$$F^1 = -\log\left(\sqrt{\mathrm{Im}\sigma\mathrm{Im}\rho}|\eta(\sigma)|^2|\eta(\rho)|^2\right),$$

where σ and ρ are Käher and complex moduli of T^2 respectively. At genus two, we can still carry out direct computation (see [1]) and derive

$$F^2(u_L,u_R) = \sum_{(n,m)\neq(0,0)} \left(\frac{u_L^1u_R^1}{n+m\sigma} + \frac{u_L^2u_R^2}{n+m\bar{\sigma}}\right)^4. \tag{3.1}$$

By expanding this expression in powers of u_L and u_R, we find that the component $F^2_{2,2}$ of this amplitude is given by the Eisenstein series of degree 4 as[2]

$$F^2_{2,2} = \sum_{(n,m)\neq(0,0)} \frac{1}{(n+m\sigma)^4}.$$

[2] This is upto an overall normalization independent of σ. To obtain the topological normalization in section 4, we need to multiply this by $\frac{1}{4(2\pi)^4}$

In the next section, we will show that this is consistent with the topological interpretation of $F^2_{2,2}$ as integration of the first Chern class c_1 of the Hodge bundle over the one dimensional space of moduli of holomorphic maps from genus 2 surfaces to T^2.

For $g \geq 3$, it becomes exceedingly difficult to compute F^g directly. There we can use the harmonicity equations, (2.7) and (2.8), to obtain F^g.

Let us first write down the harmonicity equation (2.7) on $T^2 \times R^2$ for general value of g. In terms of the components, the equation is

$$\begin{aligned} & D_{t^{22}} F^g_{n,m} - D_{t^{12}} F^g_{n-1,m} + \\ & \quad + \frac{2g-2+m}{2g-2-m+1}\Big(D_{t^{21}} F^g_{n,m-1} - D_{t^{11}} F^g_{n-1,m-1} \Big) = 0 \end{aligned} . \tag{3.2}$$

Suppose t^{22} couples to the marginal operator $\partial_z X^{\bar{1}} \partial_{\bar{z}} X^1$ where X^1 is the coordinate in the T^2 direction (namely $t^{22} = -(8\pi i)^{-1}\bar{\sigma}$). In this case, t^{12}, t^{21} and t^{11} couple to $\partial_z X^2 \partial_1 X^1$, $\partial_z X^{\bar{1}} \partial_{\bar{z}} X^{\bar{2}}$ and $\partial_z X^2 \partial_{\bar{z}} X^{\bar{2}}$ respectively. In this case, it is easy to see that only nontrivial case in (3.2) is when $n = m$, otherwise each term in the equation vanishes identically. Since X^2 is in the R^2 direction, $X^2(z,\bar{z})$ is a single valued function on the Riemann surface Σ. It is then straightforward to compute insertions of these operators in F^g and obtain

$$\begin{aligned} (t+\bar{t}) D_{t^{12}} F^g_{n-1,n} &= (2g-2+n) F^g_{n-1,n-1} \\ (t+\bar{t}) D_{t^{21}} F^g_{n,n-1} &= (2g-2+n) F^g_{n-1,n-1} \\ (t+\bar{t}) D_{t^{11}} F^g_{n,n} &= (g+n) F^g_{n,n} \end{aligned}$$

We can derive these formula by writing, for example, $\partial_z X^2 \partial_1 X^1 = \partial_z \Big(X^2 \partial_1 X^1 \Big)$ and by doing integration-by-parts. By substituting them into (3.2), we obtain

$$(t+\bar{t}) D_{\bar{t}} F^g_{n,n} = \frac{2g-2+n}{2g-2-n+1} (g-n) F^g_{n-1,n-1} \tag{3.3}$$

when $t^{22} = \bar{t} = -(8\pi i)^{-1}\bar{\sigma}$. Similarly when $t^{11} = t = (8\pi i)^{-1}\sigma$, (3.2) becomes

$$(t+\bar{t}) D_t F^g_{n,n} = \frac{2g-2-n}{2g-2+n+1} (g+n) F^g_{n+1,n+1} \tag{3.4}$$

By combining these two equations, we also find that $F^g_{n,n}$ is an eigen-function of the Laplace operator

$$(t+\bar{t})^2 D_t D_{\bar{t}} F^g_{n,n} = (g-n)(g+n-1) F^g_{n,n}. \tag{3.5}$$

When $g = 2$, the harmonicity equations, (3.3) and (3.4), gives

$$\begin{aligned} & D_{\bar{t}} F^2_{2,2} = 0 \ , \qquad D_{\bar{t}} F^2_{1,1} = \frac{3}{2} F^2_{0,0} \\ & D_t F^2_{1,1} = \frac{3}{4} F^2_{2,2} \ , \quad D_t F^2_{0,0} = \frac{4}{3} F^2_{1,1}. \end{aligned}$$

It is straightforward to check that they are consistent with the explicit expressions (3.1) for F^2. Now we can apply the harmonicity equations, (3.3) and (3.4), to compute F^g for all g.

Let us now use the harmonicity equations to determine F^g for all $g \geq 3$. By combining (3.3) and (3.4) with the hermiticity condition $\overline{F^g_{n,n}} = F^g_{-n,-n}$, we find $F^g_{n,n}$ $(\sqrt{dt})^{g+n}$ $(\sqrt{\bar{dt}})^{g-n}$ is invariant under the duality transformation. It is straightforward to show that all $F^g_{n,n}$ have finite $t, \bar{t} \to \infty$ limit. In fact $F^g(u_L, u_R)$ should approach (const) $\times\, (u_L^1 u_R^1 + u_L^2 u_R^2)^{4g-4}$ in this limit. By imposing the duality invariance, We find that (3.5) has a unique solution for $F^g_{0,0}$ as

$$F^g_{0,0} = (\mathrm{const}) \times \sum_{(n,m)\neq(0,0)} \frac{1}{|n+m\sigma|^{2g}}.$$

We can then use (3.3) and (3.4) to compute the rest of $F^g_{n,n}$ to obtain

$$F^g(u_L, u_R) = (\mathrm{const}) \times \sum_{(n,m)\neq(0,0)} |n+m\sigma|^{2g-4} \left(\frac{u_L^1 u_R^1}{n+m\sigma} + \frac{u_L^2 u_R^2}{n+m\bar{\sigma}} \right)^{4g-4}.$$

4. Topological Interpretation

Given the fact that the physical $N = 2$ string amplitudes have been reformulated in terms of topologically twisted $N = 4$ theories, it is natural to ask if there is any topological meaning to the latter. Recall that if we have any $N = 2$ superconformal theory we can consider the twisted version and couple it to topological gravity, which has critical dimension 3. The geometrically interesting examples of such theories are sigma models on Calabi-Yau manifolds and depending on how the left- and right-moving degrees of freedom are twisted we get a topological theory which counts holomorphic maps (A-model) or quantizes the variations of complex structure on the Calabi-Yau (the Kodaira-Spencer theory [5] obtained from B-model). If the complex dimension of Calabi-Yau is not equal three the topological string amplitude vanishes because the $(3g-3)$ negative charges of the G^- insertions is not balanced by the $d(g-1)$ charge violation of the $U(1)$ of the $N = 2$ algebra if $d \neq 3$. Only in the case of complex dimension 2 can still try to get a non-vanishing amplitude by inserting $(g-1)$ chiral operators to the action of the form $G_L^- G_R^- V$ where V is the unique chiral field with charge two[3] and it corresponds to the volume form of the complex 2d manifold. Note using (2.3) that these $(g-1)$ insertions are the same as the $(g-1)$ insertions of $\tilde{G}_L^+ \tilde{G}_R^+$. In other words it gives exactly the same result

[3] In dimension bigger than 3 we need a negative charged chiral field which does not exists, and in dimension 1, there is no chiral field with charge bigger than one.

as the partition function for the highest instanton number of the $N=2$ string (2.5) with the exception of the insertion of $\int J_L J_R$. It was argued in [5] that this $N=2$ topological amplitude vanishes even with this charge insertion. In fact it was directly argued in [4] that this follow rather simply from the underlying $N=4$ algebra. So the $N=2$ string amplitude manages to be non-trivial precisely because of the extra insertion of $\int J_L J_R$. Therefore there must be a simple topological meaning for the highest instanton amplitude of the $N=2$ string.

For concreteness let us consider the A-model version which is set up to count the holomorphic maps from Riemann surfaces to Calabi-Yau manifolds. In the limit that $\bar{t}_i \to \infty$ one can show that the measure is concentrated near the holomorphic maps [5] . In this case we are considering holomorphic maps which map the Riemann surface with $(g-1)$ points on the Riemann surface mapped to specific $(g-1)$ points on the target which is dual to the volume form. Actually to go to the Poincare dual to the volume form one has to use G^+ trivial operators to deform the field, but that may change the amplitude in this case because we have J insertion which does not commute with G^+. So we have to use the precise representative given by $G_L^- G_R^- V$. Each time we choose a cohomology representative in target of degree d (corresponding to d-forms), it gives rise to a $(d-2)$ form on moduli space (which translating degree to charge, in the operator language means that the charge is decreased by two units because of the insertion of $\oint G_L^- \oint G_R^-$)[4]. In our case each volume form will give a $(1,1)$ form on the moduli space of holomorphic maps which we denote by k. So consider the moduli space $\mathcal{M}^g$ of holomorphic maps from genus g to the 2 complex manifold. The formal complex dimension of $\mathcal{M}$ is $(g-1)$, however it typically has a dimension bigger than $(g-1)$. In such cases the topological amplitude computation is done by considering the bundle $\mathcal{V}$ on $\mathcal{M}$ whose fibers are the anti-ghost zero modes which is $H^1(N)$ where N is the pull back of the normal bundle piece of the tangent bundle on the manifold restricted to the holomorphic image of the Riemann surface. Let n be the dimension of $\mathcal{V}$. Then the complex dimension of $\mathcal{M}$ is $(g-1+n)$. Therefore if it were not for the $\int J_L J_R$ insertion, the usual arguments of topological strings, in the simple cases, would lead to the computation of

$$\int_{\mathcal{M}} k^{g-1} c_n(\mathcal{V}) \tag{4.1}$$

where c_n denotes the n-the chern class of $\mathcal{V}$. However as mentioned before this amplitude vanishes. The effect of the $\int J_L J_R$ insertion, will correspond on the moduli of holomorphic maps to a $(1,1)$ form which we denote by $\mathcal{J}$. This has the effect of absorbing one of the

[4] The form on the moduli space can be described by considering the canonical map from the total space of the Riemann surface and the moduli space of holomorphic maps to the target manifold, and using the pull-back of the d-form and integrating it over the Riemann surface.

fermion zero modes which was responsible for the vanishing of the amplitude. Thus the characteristic class that we will end up with from $\mathcal{V}$ will be of dimension $(n-1)$. The precise form of it may depend on the case under considerations. Therefore using the same reasoning as for topological theories we see that the top instanton number amplitude for $N=2$ strings in the $\bar{t}\to\infty$ computes

$$F^g_{2g-2,2g-2}\big|_{\bar{t}\to\infty}=\int_{\mathcal{M}^g} k^{g-1}\wedge c_{n-1}(\mathcal{V})\mathcal{J} \tag{4.2}$$

Later in this paper we will see how this works in detail in the case of the four manifold $T^2\times R^2$ for $g=1,2$. It happens that for some topological strings the formula (4.1) is modified. An example of this is discussed in [9]. In such cases some of the insertions of operators corresponding to fields (the analog of k in the above) will be replaced by modifying the bundle V. It turns out that this does happen for us for $g\geq 3$ for the example of $T^2\times R^2$. In particular for $g>3$ the above formula in this case gets replaced by

$$F^g_{2g-2,2g-2}\big|_{\bar{t}\to\infty}=\int_{\mathcal{M}^g} k\wedge c_{n+g-3}(\tilde{\mathcal{V}})\mathcal{J} \tag{4.3}$$

for some $\tilde{\mathcal{V}}$. Unfortunately there is no general prescription for computing this that we are aware, and it very much depends on the models. We have not computed $\tilde{\mathcal{V}}$ in our case.

Let us now turn to the specific case of $T^2\times R^2$. In the case of genus one the above computation is exactly the same as counting the holomorphic maps from torus to torus, because the $\mathcal{J}$ insertion precisely absorbs the zero mode in the direction of R^2 and so we are back to counting holomorphic maps from genus one to genus one, which was done in [5].

For genus g, the moduli of holomorphic maps $\mathcal{M}$ has dimension $(2g-2+1)$ for degree bigger than zero. This corresponds to double covering of the torus by the Riemann surface having $(2g-2)$ branch points and $(+1)$ comes from choice of the R^2 coordinate of the holomorphic map. Note that all holomorphic maps to $T^2\times R^2$ will lead to constant maps as far as the R^2 factor is concerned. Thus pulling back the volume form V and integrating over the Riemann surface will lead us to the statement that k is precisely the $(1,1)$ form on $\mathcal{M}$ in the direction of changing the R^2 image. The bundle $\mathcal{V}$ in our case is the same as holomorphic one forms, simply because the normal bundle is simply the R^2 direction (i.e. the fermion zero modes in the R^2 direction). In other words $\mathcal{V}$ is simply the Hodge bundle $\mathcal{H}$ on the moduli of genus g surfaces, restricted in our case to the moduli of Riemann surfaces which holomorphically cover a fixed torus. The top chern class is g, but we are instructed to take the $(top-1)$ class, which is $c_{g-1}(\mathcal{H})$. Note that the dimension of $\mathcal{M}$ agrees with $(g-1)+(g-1)+1$ as expected from (4.2).

Let us first consider the case of $g=2$. In this case we are instructed to compute $\int k\wedge c_1\wedge\mathcal{J}$ over the moduli space of holomorphic maps which is of dimension 3; 2 coming from the choices of two branch points and 1 from the image of the map on R^2. As discussed about the k integrates over the R^2 part and gives the volume in the R^2 direction. Moreover $\mathcal{J}$ gives the volume form over the torus, i.e. absorbs the zero mode corresponding to shift of the origin of the map on the torus direction. Note that if we did not have $\mathcal{J}$ and if we have c_2 instead of c_1 the computation would have been the standard $N=2$ topological computation which would have vanished because of the flatness of the torus. This agrees with the general argument that the J insertion is crucial for a non-vanishing answer. We are thus left with $\int c_1$ over the moduli of holomorphic maps from genus 2 to torus, up to a shift in the origin of the torus.

In the previous section, we have seen that the top component $F^2_{2,2}$ of the genus 2 amplitude is holomorphic in t and is given by the Eisenstein series E_4. On the other hand, the topological interpretation discussed here suggests that $F^2_{2,2}$ is the same as integration of the first Chern class c_1 of the Hodge bundle over the one dimensional space of moduli of holomorphic maps. Moreover using other argument we have shown that the top component is proportional to E_4. We will now prove that the answer being proportional to E_4 could have also been derived using the direct topological computation.

To this end we have to use the fact that c_1 that in genus 2 can be written as

$$c_1 = 2\pi i \partial\overline{\partial}\log\det\mathrm{Im}\Omega$$

and try to use integration by parts to integrate over moduli of holomorphic curves. However in order to do this we cannot use the above expression because det $\mathrm{Im}\Omega$ is not a modular invariant object. Instead we write it as[5].

$$c_1 = \frac{1}{2\pi i}\partial\overline{\partial}\log\Big[\det\mathrm{Im}\Omega\big(\prod_{even\ \theta\ functions}\vartheta\bar{\vartheta}\big)^{1/5}\Big]$$

which is modular invariant. Note that product of even θ functions has no zeroes in the interior of the moduli space for $g=2$ (a fact that fails to be true for higher genera). Since we have a total derivative we can integrate by parts and we thus come to the point on the moduli of holomorphic maps which corresponds either to a handle degeneration or to splitting to two genus 1 curves. The product of even theta functions in the handle degeneration case has a zero of the order $z^{1/2}$ and in the case of splitting a zero of the order z. So in order to compute $\int_{\mathcal{M}} c_1$ we simply have to count how many holomorphic curves exist which go from a handle degenerated genus 2 to torus and multiply it by 1/10 and add to it the number of holomorphic curves which exist when we have the splitting case and multiply it by 1/5. This is described mathematically by the statement that

$$c_1 = \frac{1}{10}(2\delta_1 + \delta_0) \tag{4.4}$$

[5] Which is the same trick that give the 2 loop bosonic string amplitude [10]

where δ_1 denotes the first chern class of a bundle whose divisor is the boundary of moduli space corresponding to genus 2 splitting to two genus 1 curves and δ_0 denotes the the corresponding one where the divisor is the boundary of moduli space where the genus 2 curve has a handle degeneration. Note that we have chosen coordinates on the moduli space such that a symmetry factor of 1/2 in the δ_0 and δ_1 degenerations are included.

Using (4.4) we are in a position to compute the genus 2 topological amplitude in terms of genus 1 amplitude[6]. First note that a genus 2 covering of a torus will lead to two branch points. The degenerate genus 2 curves can occur only when the two branch points collide. Not every colliding branch points give rise to degenerate Riemann surfaces, as some of them simply convert 2 branch points of order 2 to a single one of order three. Those would not contribute to our amplitude. To count the degenerations of the other type, note that if you remove the degenerate preimage we end up in the handle degeneration case to a holomorphic map from torus to torus where we have marked two of the covering sheets (the ones which get glued over the handle degeneration) and in the splitting case to two genus one curves connected by a tube. In the handle degeneration case if the remaining genus 1 to torus map is of degree n, we have $n(n-1)/2$ ways to choose the sheets, and so putting all the contributions of these together, and denoting the genus 1 answer by F_1 (the topological part of it which is $dF_1/dt = \eta'/2\pi i\eta$) we see that the handle degeneration gives (noting that 1/2 is already counted in the definition of δ_0)

$$\frac{1}{10}\cdot\Big[\frac{d^2F_1}{dt^2}-\Big(\frac{dF_1}{dt}+\frac{1}{24}\Big)\Big]$$

(note that each d/dt gives a factor of n–note that since we are in the topological limit of $\bar{t}\to\infty$ we do not have covariantization of d/dt). We have added $+1/24$ to dF_1/dt to eliminate to degree zero part of the map which we take into account separately below. Similarly when we get the splitting case we get two maps from two different genus 1 curves to our torus. We simply have to choose a sheet from each one to identify with the other. If one of them is a covering of order n and the other of order m, we get nm ways of doing this. We also have to divide by a symmetry factor of 1/2 because of the Z_2 symmetry of exchanging the two genus 1 curves. We thus get a contribution from the splitting case (noting that the symmetry factor 1/2 is already included in the definition of δ_1)

$$\frac{1}{5}\cdot\Big(\frac{dF_1}{dt}+\frac{1}{24}\Big)^2$$

In addition to these two contributions we have bubbling type contributions, which correspond to degenerate maps from a genus 2 to the torus, where the genus 2 curve is itself a torus glued to another torus, where one torus gets mapped to a constant, and the other

[6] We are grateful to R. Dijkgraaf for explaining this to us.

gets holomorphically mapped to the torus. The c_1 of this family will simply be the c_1 of the genus 1 curves times the one point function of the genus 1 answer. Since c_1 on the genus 1 moduli space gives 1/12, the bubbling contribution is given by

$$\frac{1}{12}(\frac{dF_1}{dt}+\frac{1}{24})$$

There is also going to be an overall constant contribution coming from genus 2 curves which map to a constant. Using the topological formula (4.2), and the fact that in this case $\int k\wedge\mathcal{J}$ absorb the volume integral over $T^2\times R^2$, this should be $c_3(\mathcal{H}\oplus\mathcal{H})=2c_1(\mathcal{H})c_2(\mathcal{H})$ and using the fact that $2c_2=(c_1)^2$ it is given by $(c_1)^3$. Integrated over moduli of genus 2 curves, this gives $\frac{1}{2880}$. Putting all these three contributions together we find

$$\frac{1}{10}\cdot[\frac{d^2F_1}{dt^2}-(\frac{dF_1}{dt}+\frac{1}{24})]+\frac{1}{5}\cdot(\frac{dF_1}{dt}+\frac{1}{24})^2+\frac{1}{12}(\frac{dF_1}{dt}+\frac{1}{24})+\frac{1}{2880}$$

It is quite miraculous that all the terms which are not second order in derivatives of t disappear as they should in order to end up with a function of a definite modular weight. Moreover E_4 which was shown to be proportional to the genus 2 answer is proportional to $\frac{d^2F_1}{dt^2}+2\frac{dF_1}{dt}$ as expected. We thus learn that

$$F^2_{2,2}=\frac{1}{2880}E_4=\frac{1}{10}(\frac{d^2F_1}{dt^2}+2(\frac{dF_1}{dt})^2)$$

If we consider $g>2$ the above topological computation formally vanishes, because we get a higher power of k. Since all of them are in the direction of R^2, and there is only one such direction on the moduli space and if the topological amplitude were give by the above formula, we would get zero. In fact this is precisely an example of the type mentioned in the above, where the extra insertions go to modifying the bundle $\mathcal{V}$. This is clear from the explicit attempt in computation of the amplitude for $g>2$ because then we can no longer replace the fermion fields by the zero mode wave functions–some of the fermions are contracted, giving us Greens functions, which are reinterpreted as curvature of a bundle, as in [9]. the amplitude vanishes. In such a case presumably methods similar to those of [9] should be applicable to determine the new bundle $\tilde{\mathcal{V}}$ which we expect to be of rank $(2g-2)$, and for which the amplitude can be written as

$$F^g_{2g-2,2g-2}\big|_{\bar{t}\to\infty}=\int k\wedge c_{2g-3}(\tilde{\mathcal{V}})\mathcal{J}$$

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