Large N in $D > 2$

Michael R. Douglas
Department of Physics and Astronomy, Rutgers University
Piscataway, NJ 08855-0849, USA
E-mail: mrd@physics.rutgers.edu

ABSTRACT

We suggest some directions for future work on large N matrix and gauge field theories which might eventually allow studying issues of locality and causality in fundamental string theory. We also give a brief overview of recent work on the large N limit of the exact solution of four-dimensional supersymmetric gauge theory.

1. Introduction

A lot of effort, over more than twenty years, has gone into trying to solve field theories (say in D dimensions) and lattice models of $N \times N$ matrix variables $M(x)$ or $A_\mu(x)$ in the limit $N \to \infty$, or as an expansion in $1/N$. (Fairly extensive bibliographies can be found in Refs 1-3.) Such theories have well-known connections to string theory. The first were reformulations of gauge theories as sums over surfaces, the simplest being the strong coupling expansion on the lattice, which might have continuum limits producing novel string theories (in D dimensions). Another approach is to regard weak coupling (Feynman diagram) expansions as discretizations of two-dimensional gravity. Using the double scaling limit, these can be used to construct fundamental string theories, typically non-critical strings in target space-time dimension $D + 1$.

In both cases, the leading large N limit becomes genus zero in the string perturbative expansion, and $1/N$ becomes the closed string coupling constant: string reformulations are perturbative expansions in $1/N$. We can turn this around: given a string reformulation, we can regard the original functional integral (or a scaling limit) as a non-perturbative definition of the same string theory. The hope that this can be done for fundamental superstring theory is perhaps the best reason to be interested in these models.

However, for $D > 1$ these field theories are much harder to treat than the original world-sheet formulation. Our only real hope of getting fundamental string physics out of them is to identify qualitative features which we can understand without exact solutions. Clearly we should look at the most distinctive features of string theory. A good example is the generic $e^{-1/\sqrt{\hbar}}$ behavior of non-perturbative effects. [4]

2. Locality and Causality in String Theory

Surely the most distinctive feature of string theory is that it is a theory of extended objects. It is widely appreciated that this will ultimately lead us to a concept of space-time radically different from that of field theory and in particular general relativity.

Let us start with a simpler problem. Our understanding of geometrical field theories such as Yang-Mills and general relativity starts with an understanding of the classical field theory. Although the quantum theory can be very different, it is best to use geometric concepts such as metric, connection, and general covariance in formulating and understanding it, and their definitions are classical. Perhaps the same approach will be fruitful for string theory – to restrict ourselves to classical string field theory, understand its geometry, and develop new concepts of locality and causality there. Of course one can argue for and against this idea; for example very interesting new features such as topology change already appear at genus zero in studying string compactification, but on the other hand features such as duality are only true of the quantum theory. In any case, it would seem simpler to study the classical theory first.

In the context of large N and matrix models, a good reason to make this simplification is that it corresponds to taking the $N \to \infty$ limit of the field theory first, which produces a real simplification. For definiteness (this will suffice to make my main points) let us consider a zero-dimensional model with only two matrices A_{ij} and B_{ij}, an integral with an action $S[A, B]$ which is a general function invariant under global $U(N)$ rotation $A \to UAU^{-1}$ and $B \to UBU^{-1}$. The observables relevant for string theory (and gauge theory) are also $U(N)$ invariant and these satisfy factorization, $\lim_{N\to\infty} \langle \text{tr } W_1 \text{ tr } W_2\rangle = \langle \text{tr } W_1\rangle\langle \text{tr } W_2\rangle$. Here W_1 and W_2 are any 'word' formed from the letters A and B such as A^n, B^n, $A^m B^n$, $\ldots$. Two words not related by a cyclic permutation give different invariants, and their expectation values are essentially independent in the limit $N \to \infty$: one can vary them independently with suitable variations of the action. More generally, the words would be arbitrary sequences of points in space-time. In a gauge theory we would consider only the gauge-invariant Wilson loops.

The (old) idea is that the limit is a new 'classical' field theory, different from the original $\hbar \to 0$ limit. The expectation values $\langle \text{tr } W\rangle$ give a complete set of coordinates on its configuration space, and using factorization, the Schwinger-Dyson equations become the new classical equations of motion. For the two-matrix model, we have (for each W and for B as well)

$$\langle \text{tr } \frac{\partial S}{\partial A} W\rangle = \frac{1}{N} \sum_{W=W_1 A W_2} \langle \text{tr } W_1\rangle \ \langle \text{tr } W_2\rangle. \tag{1}$$

All $\langle \text{tr } W\rangle$ are $O(N)$ and this non-linear equation contains enough information to determine the limit of all vev's in the field theory.

In all the existing string interpretations of large N field theories, we can make a direct correspondance between small $O(N^0)$ fluctuations of the configuration $\delta\langle \text{tr } W\rangle$ around a solution, and one-string states. Further terms in an expansion of the solution in $1/N$ can be interpreted as sums over genus zero world-sheets. Thus this formalism produces a non-perturbative definition of the classical limit of the string theory. Although the words 'non-perturbative' and 'classical' might seem contradictory, they are not – what one means is that one has a full non-linear equation of motion valid for arbitrarily strong fields. These ideas apply to any large N field theory with adjoint and vector fields.

A notation which emphasizes the similarity to field theory is to let the collection of expectation values define a 'string field' $\phi(W) = \langle \mathrm{tr}\ W \rangle$, and rewrite (1) as

$$\mathcal{L}\phi = g_{st}\phi * \phi. \tag{2}$$

If $\phi_i(x)$ were a finite set of functions (and we allowed general coupling constants g_{ijk} on the right) this would be a generic non-linear PDE.

In comparing this to continuum string field theory,[5] the most notable simplification is that we avoided the complicated infinite set of interactions required to reproduce the usual integrals over closed string moduli space. What makes this possible is that space-time covariance is not fully manifest; one of the space coordinates (the Liouville zero mode of non-critical string) is represented by the length of a word, or in continuum terms has been used to specify a world-sheet coordinate system. I believe it is fair to say that this correspondance has not been completely understood, even in the tractable $c = 1$ case. One advantage of studying locality and causality in $D > 2$ is that there are transverse dimensions, which are easier to interpret. Furthermore, in the most sensible picture of non-critical string theory, the Liouville zero mode is a space-like coordinate. Thus we expect that if two points are at space-like separation in D dimensions, they will be at space-like separation in $D + 1$ dimensions.

Let us suppose we have a matrix model which we believe has a scaling limit which produces an interesting higher dimensional string, ideally a superstring theory. We can try to use these ideas to study locality and causality in this theory. We have a precise configuration space, and the factorized Schwinger-Dyson equations provide a classical equation of motion.

The equation (1) is a rather formal description of the theory and as a warm-up for thinking about it let's suppose someone handed us the equations of motion for various field theories including gravity and matter, written in component form with no further explanation. We would first observe that they are non-linear second order PDE's with some obvious solutions, such as flat space. We would go on to study how a small variation of the fields propagates, and eventually discover the idea of light-cone. The essential points for understanding causality are that this can be defined locally (in an arbitrarily small but finite region) and it depends only on the metric. Mathematically, the characteristics of a PDE are generally determined by the highest derivative term.

We might next try to repeat this exercise for a component expansion of the string field theory Lagrangian. We would quickly realize that since the coupling between a component field of spin s and two scalars involves derivatives of order s, there is no single component field which determines the causal structure. In this sense, the string field equation of motion is fundamentally different from a PDE and thus from all previous classical field theories. So far, nobody knows how to pose the standard initial-value problem, with boundary conditions given on a space-like surface, in string theory.

It is important to realize that these are non-perturbative questions. At any finite order in perturbation theory, varying the background does not change the causal structure. The point at which this is determined is in the choice of which linear operator to invert to define the propagator. To see a change of causal structure, one must include some interaction with the background in this operator.

Now matrix models do contain answers to these questions. For example, there is no difficulty in defining the solution with specified boundary conditions on a space-like surface, because it was defined in the original field theory. What makes these answers difficult to get at is that we don't have much control over the scaling limit or the precise map between scaled results and continuum results in $D > 2$.

The main point I want to make is that we probably don't need such control to get interesting results. It would be very interesting to know how causal structure can be modified by the choice of background in any $D > 2$ matrix model, even before the scaling limit. Perhaps ideas relevant for string theory can be tested with matrix models without having a full correspondance. How we do this depends on what we believe the correct idea of locality and causality in string theory might be. Different formalisms suggest different ideas. Perhaps if we avoided expanding in component fields, we would discover more appropriate ideas of geometry in loop space, or perhaps there just is no locality at distances shorter than the string scale, and we need to find the new idea which replaces it.

Perhaps with such an attitude we can avoid the intractable problem of solving equations such as (1). Unfortunately, we have no general techniques for approximating the solutions, or even writing down ansatzes for the solutions.

If we think about how we might do this in field theory, we see a major shortcoming in our understanding of large N field theories and string theory. For all but the most primitive work on non-linear PDE's, it is essential to have a good understanding of related linear PDE's and some ability to solve their initial value problem with general boundary conditions. The most important related linear PDE is the linearization of (2) with $\phi = \frac{1}{g_{st}}\phi_0 + \delta\phi$,

$$\mathcal{L}\delta\phi = \phi_0 * \delta\phi + \delta\phi * \phi_0. \tag{3}$$

Others also arise, for example the Lax pair of operators for integrable PDE's. Unfortunately, we do not know much more about linear equations for loop functionals ϕ_0 and $\delta\phi$ than about non-linear equations.

A very simple example of a linear problem is in recent work by Neu and Speicher,[6] who propose a generalization of Wegner's model of localization. This is a model of non-interacting electrons moving on a fixed lattice in a random potential. Large N

is brought in by taking N-component electron wave functions and using independent $N \times N$ hermitian matrices for the random potential.

Neu and Speicher consider the spectrum and Green's functions of the random operator

$$H = \Delta(x, y)\delta_{ij} + V_{ij}(x)\delta(x - y) \qquad (4)$$

where x and y are lattice sites and $\Delta(x, y)$ is the lattice laplacian (hopping matrix). The $V_{ij}(x)$ are independent random matrices with distribution given by integrals with an arbitrary $U(N)$-invariant weight $\exp -N \sum_x \mathrm{tr}\ w(V(x))$.

This is not a hard model to solve – using diagrams, one must sum rainbow and ladder diagrams, and the string theory analog would be an open string propagating in a background. But the main point of Neu and Speicher's work is that the recent advances initiated by Voiculescu [7] relating large N to free probability theory make this model trivial to solve. It is easy to construct a master field $\hat{V}(x)$ which reproduces the large N limit of all expectations $\langle \mathrm{tr}\ W \rangle$ of words made from the $V(x)$. This turns the problem of finding the average spectrum of the random operator H into the problem of finding the spectrum of a single Schrödinger operator $\hat{H}$ with In terms the concepts discussed by Gross in his talk here, Δ and $\hat{V}$ are relatively free operators, so the spectrum of $\hat{H}$ is simply the additive free convolution of their spectra.

The work of Gopakumar and Gross, myself, and others precisely defining master fields [1,2,8] opens the direction of extending such techniques to more interesting large N theories. However the field is still in an early stage and it is not yet clear how this work helps to solve (1) for more complicated models. I believe the appropriate next step is instead to solve more sophisticated linear problems, such as (3) with more complicated backgrounds ϕ_0, or equivalent master field problems analogous to the one of Neu and Speicher.

In the context of string theory, the main reason such problems have not been much studied is that we are used to the idea that in string theory we should not study fluctuations around a background which does not solve the equations of motion. This restriction makes locality and causality very obscure, because we can't vary the background at a single point in space-time. Such backgrounds are completely well defined in a large N field theory before we take the scaling limit, making these issues much more accessible.

We should start by asking, since we are starting from a field theory, how could we get a different causal structure from the original field theory? Perhaps the formalism of loop equations is unnecessary – after all, if we can sum the standard weak coupling expansion, we get a solution of the equations. The sum of planar diagrams has finite radius of convergence (assuming we have both UV and IR cutoffs) so this sounds like a valid approach, and it sounds like this must reproduce the original concept of space-time as the underlying field theory. We could think of the strings as assemblies of particles (or 'beads') moving in the original space-time.

This is probably not the whole story. Suppose we make a finite perturbation of the vev for a loop L of finite extent, $\phi_1(W) = \phi_0(W) + c\delta(W, L)$. (The 'delta function' $\delta(W, L)$ is a functional of W with support on loops 'near' L.) The linearized propagation around such a background now includes the interaction with this loop, $\delta\phi * \delta(W, L)$, and this is non-local in terms of the original space-time. This is certainly one natural way to generalize the concept of causal structure for string theory, and is worth studying.

It is also important to realize that the weak coupling expansion might not be summable. A good way to see this is to think about the strong coupling expansion for a gauge theory. It has finite radius of convergence and so formally it is summable as well. Now the two expansions have very different qualitative properties at leading order, and it would be highly non-trivial for both to sum to the same results. In fact they do not in almost every solvable case – the $N \to \infty$ limit exhibits phase transitions as a function of the coupling. It is a good bet that this is true in general, because one can understand the transition as a consequence of positivity constraints on physically realizable configurations. [1] These include the constraint $\rho(\lambda) \geq 0$ for the spectral density of each of the matrix variables, and multi-matrix generalizations. They are additional structure of configuration space not already contained in (1), and often must be imposed separately to obtain correct solutions. One can see from known properties of $D > 2$ gauge theory that in the two limits, different constraints are expected to be saturated. Thus the continuation of the solution in one limit will not produce the other limit, and both of the standard expansions fail.

It is likely that this structure is relevant for matrix models of fundamental strings as well, and this can be illustrated with the $c = 1$ matrix model. The standard treatment describes the large N limit of the phase space with a spectral density $\rho(\lambda)$ and conjugate variable $\Pi(\lambda)$. Suppose we were unable to solve for the (trivial) collective field theory ground state. We might make the following preliminary observation: since the constraints become important where $\rho(\lambda) = 0$, it would be interesting to study a point in λ space separating a region in which $\rho(\lambda) > 0$ from $\rho(\lambda) = 0$. Of course the true ground state has such a point and indeed the scaling limit focuses on it. Perhaps classifying the analogous possibilities in $D > 2$ will allow us to identify possible scaling limits without solving the theory.

This detailed structure of configuration space may be important, but the most important similarity between large N field theory in $D > 2$ and string theory surely is the size of the configuration space. At the most naive level, since both theories are formulated in terms of loop functionals, we would say they are comparable. However there is also a sense in which the large N configuration space is far larger than the configuration space of classical string field theory!

The simplest measure of this is to consider a 'grading,' an integer n which allows us to divide the invariants into finite sets S_n, and ask how the size of the set grows with n. In D-dimensional free quantum field theory, the Hilbert space has a basis of one-particle, two-particle states and so forth, and the energy $\mathcal{E}$ provides a natural grading. It is familiar that the number of states grows roughly as $\exp \mathcal{E}^{1-1/D}$.

Small variations of the loop functionals $\Phi[X(\sigma)]$ of string field theory are in correspondance with states of the two-dimensional world-sheet theory and in this grading (using $\mathcal{E} = L_0 + \bar{L}_0$) the number of states grows as $\exp \mathcal{E}^{1/2}$. This is 'world-sheet energy,' related to space-time mass as $\mathcal{E} = m^2$, so this subexponential behavior is compatible with the familiar Hagedorn behavior $\exp m$. For many purposes (in particular if one tries to precisely define the operations in the string field theory Lagrangian) the world-sheet density of states is the more relevant measure of this size.

In large N field theory, a natural grading is the length L of the word, and the number of words grows as $\exp L$. It appears that this is the appropriate measure of the 'size' of the configuration space (it governs the structure of the terms in (1)), and in this sense the size is larger than any quantum field theory.

The difference between sub-exponential and exponential growth appears to be crucial and apparently even the mathematicians have only a primitive understanding of such spaces and the related operator algebras. A very elementary introduction to this is given in Ref. 1, with references for further reading.

In any case I want to emphasize the large number of degrees of freedom as a similarity between large N field theory and string theory. At this point the reader may be shaking his head, as it has become almost a dogma in recent years that string theory (in $D > 2$) has *fewer* physical degrees of freedom than a local quantum field theory. This is an important idea and I will not try to review the arguments which are given to support it, but instead propose that in a different and more formal sense, string theory has more degrees of freedom than field theory. Indeed the two statements might not be in conflict! Let me draw an analogy to the relation between classical and quantum mechanics. In general, a physicist would say that a quantum system has fewer degrees of freedom than its classical limit. This is clearest in the thermodynamics as the states with $E >> kT$ are frozen out and of course this observation in the context of black body radiation was the original motivation for quantum mechanics.

However, there is also a familiar mathematical sense in which a quantum mechanical system has more 'degrees of freedom' than its classical limit. If we ask how much mathematical data we need to give to completely specify the state of a particle (say in one dimension) at a given time, we need a wave function $\psi(x)$ as compared to two numbers (x, p). This is much more data, and it is unavoidable – to do quantum mechanics we must allow superposition of states.

I believe this analogy will turn out to be valid – string theory has fewer 'physical degrees of freedom' than field theory, but any complete formalism will require more 'mathematical degrees of freedom,' at least the number already visible in conformal field theory and string field theory. But time will tell.

3. Supersymmetric Gauge Theory

The reader has probably gathered from my choice of topics that I suspect that large N QCD will not be solved anytime soon, and that although a correct and useful string reformulation may exist, discovering it will require new ideas.

A traditional, less ambitious approach to large N QCD is to study its qualitative properties, and where necessary simply assume that it is qualitatively similar to QCD with $N = 3$. (The classic introduction is Ref. 9.) The assumption stressed in the past was confinement – all poles in Green's functions correspond to propagation of color singlet objects. This includes the assumption that the mass gap stays finite as $N \to \infty$. Combining this with general results such as the topological classification of Feynman diagrams leads to many consequences. The most famous example is the explanation of the Zweig rule as due to the suppression of internal quark loops by $1/N$. It was also argued that mesons should be metastable with a three-meson coupling $O(1/N)$, exotic bound states are absent, solitons of the meson theory can be interpreted as baryons, and so forth. More recent work has made quantitative statements as well, for example Refs. 10.

Recent work on supersymmetric gauge theory[11,12,13] has provided the exact low-energy effective field theory for $\mathcal{N} = 2$ and 'almost $\mathcal{N} = 2$' gauge theory in four dimensions! Now this does not (yet) directly answer many of the questions of hadronic physics, because supersymmetric QCD is not QCD, because we are interested in finite energies, and so on, but it does allow us to make very non-trivial tests of the earlier analytic approaches. In general these approaches did not make strong restrictions on the matter content of the gauge theory, so a priori they are just as well justified for a gauge theory with the special matter content required by supersymmetry as for QCD. Of course this statement should not be accepted blindly but thought about carefully in the context of a particular approach.

In the case of large N, there is no difficulty with adding the adjoint gauginos and scalars required by $\mathcal{N} = 2$ supersymmetry – the topological classification of diagrams works in the same way. Although gaugino and scalar loops are not suppressed by $1/N$, we can still assign a definite topology to each diagram, so if the input assumptions are still true, we can test the large N lore.

The basic low energy physics of supersymmetric gauge theory was discussed in Refs. 11-13. The $\mathcal{N} = 1$ theory minimally coupled to an adjoint chiral superfield has $\mathcal{N} = 2$ supersymmetry. This theory is solvable and is always in a Coulomb phase described by an effective $U(1)^{N-1}$ gauge theory. Non-renormalization theorems guarantee the existence of flat directions in the potential for the scalar component of the chiral superfield which allow a continuous moduli space of vacua (any vev satisfying $[\phi, \phi^+] = 0$). There are quantum corrections to the masses of the charged gauge bosons and monopoles, whose effect is to eliminate the classical points of unbroken non-abelian symmetry – they split up into points at which monopoles and dyons become massless.

A small mass term for the chiral superfield breaks the supersymmetry to $\mathcal{N}=1$, but if the mass $m << \Lambda$ the gauge theory scale, this can be treated as a perturbation of the $\mathcal{N}=2$ theory. Its effect is to drive monopole condensation, which spontaneously breaks magnetic $U(1)^{N-1}$, producing electric confinement and a gap.

The light spectrum of the theory (without quarks) is thus the light spectrum of supersymmetric abelian Higgs theory, with light degenerate vector and chiral multiplets of mass $O(\sqrt{m\Lambda})$. These should be interpreted as 'glueballs' of the original gauge theory.

Such a picture of electric color confinement by monopole condensation was anticipated in the work of Mandelstam and especially 't Hooft,[15] who proposed the formal devices of introducing adjoint Higgs fields, or considering expectation values for composite adjoint scalars such as $d^{abc}\langle F^b_{\mu\nu}F^c_{\mu\nu}\rangle$, to get an effective Abelian gauge theory with monopoles.

So far it would appear that the physics of $\mathcal{N}=1$ supersymmetric gauge theory is rather similar to that of bosonic gauge theory and even QCD. Confinement is no longer an assumption but is derived (from simpler assumptions), and the large N limit of the low energy effective theory should be consistent with the old lore.

Recently Steve Shenker and I studied the $SU(N)$ effective theory at finite and large N.[14] The details are too lengthy to repeat here but besides fleshing out the picture above, we found numerous surprises! First, the explicit breaking of $SU(N)$ to $U(1)^{N-1}$ (e.g. by the scalar vev) also breaks the discrete gauge symmetry S_N, so at weak coupling (large scalar vev in the $\mathcal{N}=2$ theory) the W boson masses are non-degenerate. We found that this breaking persists even in the vacuum of almost-$N=2$ gauge theory. One consequence is that there are $(N-1)/2$ distinct string tensions for the flux tube solutions in the $N-1$ Abelian Higgs theories of the effective theory. Although the conventional definition of 'string tension' (which governs the long distance limit) picks out the smallest of these, the others are visible both in the Wilson loop expectation and in the meson spectrum. The string tensions have the rough spectrum $T_n \sim m\Lambda N \sin\frac{\pi n}{N}$, so the conventional string tension and the mass gap stay finite as $N\to\infty$, but the ratio between this and the largest string tension is $O(N)$. Thus a large hierarchy of scales is present.

These results could also be used to understand bound states of heavy quarks. In the $U(1)^{N-1}$ effective theory of confinement, each of the N different colors of quark has a different $U(1)^{N-1}$ charge. A neglected puzzle in this theory is that apparently bound states will come with multiplicity because of this – for example $q\bar{q}$ has multiplicity N. At first one might think these should be identified to produce the usual spectrum of color singlets visible in the strong coupling expansion. Perhaps surprisingly, this is not correct – rather the $q\bar{q}$ states form a 'split' multiplet of N quasi-degenerate (mass splittings of order m) metastable (the decay rate vanishes as $m \to 0$) states.

The different colors of quark are distinguishable because of spontaneous breaking of S_N discrete gauge symmetry and one might wonder if this symmetry is restored by a phase transition at some m_c, producing a 'confined S_N phase.' Often phase transitions can be ruled out in a theory with unbroken supersymmetry: the vacuum energy is always zero, so first order transitions are not possible, while second order transitions are associated with massless particles, which are not present here. Indeed, there is no order parameter distinguishing the broken and confining phases, and thus our description can be continuously connected to a more conventional description with strict color confinement.

Since the theory contains a light scalar (with bare mass m), the conventional interpretation of the split multiplets is as color singlet bound states with the scalar, and we presented evidence that the two interpretations are compatible. Our 'almost-$\mathcal{N}=2$' analysis was justified for $m << \Lambda$ but because there is no phase transition, the split multiplet will evolve smoothly into a tower of unstable bound states for large m, where the confining description should be appropriate.

The large N limit is particularly subtle because of the large hierarchy of scales. Indeed, one very quickly runs into a paradox, because the form of the effective Lagrangian is essentially determined by $N=2$ supersymmetry and the number of light charged particles. Here they are the monopoles, and the low energy gauge coupling constant is produced dynamically by evolving their beta function. There are $N-1$ monopoles, one in each magnetic $U(1)$ factor, and thus the final Lagrangian does not have any explicit N dependence, and scattering amplitudes do not obey the standard N counting rules. For example, the three 'glueball' scattering is $O(1)$, not $O(1/N)$.

Although the conclusion appears to be in direct contradiction with large N lore, it turns out that the two pictures are valid in different regimes, because the effective Lagrangian is only valid at energy scales $O(\Lambda/N^2)$, going to zero in the limit. The a priori estimate of its regime of validity is the lightest mass of a particle we integrated out to derive it. Here these are the charged gauge bosons, and again there is a large hierarchy of scales in the solution, with the lightest of these having mass $O(\Lambda/N^2)$.

There is a regime in which we can trust the effective Lagrangian and the abelian monopole condensation description. It is reached by taking $m \sim 1/N^4$ so that the mass gap $\sqrt{m\Lambda}$ falls within its regime of validity. Of course this mass gap vanishes as $N \to \infty$ so we explicitly violate the starting assumption of large N lore. There is no contradiction with the idea that the lore still holds for $m \sim N^0$.

The upshot is that in 'almost-$\mathcal{N}=2$' theory, monopole condensation and large N provide two different descriptions of the physics, with no overlap in their regime of validity. Perhaps supersymmetry and large N can be combined in a different way to exploit the advantages of both. But it is conceivable that the $O(N)$ hierarchy of scales is a general feature of the large N limit of four-dimensional gauge theory, and that the conventional lore would need to be modified.

I would like to thank Miao Li and Steve Shenker for enjoyable collaborations, Is Singer and Dan Voiculescu for discussions on the mathematics of large N, and the organizers for a truly memorable conference.

This research was supported in part by DOE grant DE-FG05-90ER40559, NSF PHY-9157016 and the A. P. Sloan Foundation.

1. M. R. Douglas, Nucl. Phys. B (Proc. Suppl.) 41 (1995) 66-91.
2. R. Gopakumar and D. J. Gross, "Mastering the Master Field," Princeton preprint PUPT-1520, hep-th/9411021.
3. A. A. Migdal, Nucl. Phys. B (Proc. Suppl.) 41 (1995) 151-183.
4. S. H. Shenker, in *Random Surfaces and Quantum Gravity*, eds. O. Alvarez, E. Marinari and P. Windey, Plenum 1991, pp. 191-200.
5. B. Zwiebach, "Closed String Field Theory: an Introduction," Lectures at 1992 Les Houches Summer School, MIT-CTP-2206.
6. P. Neu and R. Speicher, "Rigorous mean field model for CPA: Anderson model with free random variables," Heidelberg preprint HD-TVP-94-17, cond-mat/9410064.
7. D. V. Voiculescu, K. J. Dykema and A. Nica, *Free Random Variables*, AMS 1992.
8. I. Singer, 'The Master Field for Two-Dimensional Yang-Mills theory.'
   M. R. Douglas, Phys. Lett. B344 (1995) 117-126.
   M. R. Douglas and M. Li, Phys. Lett. B348 (1995) 360-364.
9. S. Coleman in *Aspects of Symmetry*, Cambridge 1985.
10. R. Dashen, E. Jenkins and A. V. Manohar, PUPT-1505, UCSD/PTH 94-18, hep-ph/9411234;
    M. A. Luty and J. March-Russell, Nucl.Phys. B426 (1994) 71-93.
11. N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19.
12. P. C. Argyres and A. E. Faraggi, Phys. Rev. Lett. 74 (1995) 3931-3934.
13. A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, Phys. Lett. B344 (1995) 169-175.
14. M. R. Douglas and S. H. Shenker, Rutgers preprint RU-95-12, hep-th/9503163, to appear in Nuclear Physics B.
15. G. 't Hooft, Nucl. Phys. B190 [FS3] (1981) 455.