DUALITY AND BRAIDS IN N=2 YANG-MILLS THEORIES

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ABSTRACT

We briefly report the general form of the electromagnetic duality group Γ_D for an arbitrary $N=2$ rigidly supersymmetric $SU(r+1)$ gauge theory, which extends previous explicit constructions for the $r=1,2$ cases. The results are obtained by a method that had proven useful in the past to study Calabi-Yau moduli spaces for more than one parameter, and exploits the relation between monodromy and braid groups for algebraic surfaces .

A better grasp on nonperturbative aspects of supersymmetric gauge theories has been reached after the observation that the quantum moduli space for the abelian phase of a G invariant $N=2$ rigid gauge theory can be characterized in terms of a class of genus $r = rank\ G$ hyperelliptic Riemann surfaces Σ_r, parametrized by r complex moduli, and informations on the electromagnetic duality symmetries of the theory can be extracted from the monodromy group of Σ_r [1−3].

More precisely, the electromagnetic duality group Γ_D can be inferred from the monodromy of the symplectic vector $V = (X^A, F_A)$, whose components can be viewed as the periods of the holomorphic one-form[1]

$$\omega = X^A \alpha_A + F_A \beta^A \ , \tag{1}$$

along a basis (α_A, β^A) of the $2r$ homology cycles of Σ_r.

A deeper understanding of the relation between duality and monodromy, together with important clues on how to generalize the above theory to the gravitationally coupled case and apply it to superstrings[4−17], relies on the properties of N=2 supersymmetry, which seems to present just the right amount of complication. Although it is more subtle than N=4[18] because of the presence of both perturbative and non-perturbative phenomena, at the same time it still presents a particularly restricted structure, crucial towards its solvability, that is known as "special geometry"[19−21].

The dynamics of an $N=2$ theory is fully encoded in the holomorphic prepotential $F(X^A)$[19], function of the moduli coordinates X^A , $A = 1, \dots, r$. While perturbative monodromies derive from the unique one-loop perturbative correction[22] to $F(X)$, non-perturbative monodromies, associated to points where monopoles or dyons become massless, correspond to an infinite sum of instanton contributions.

It was recognized[4] that the monodromy group $\Gamma_2 \in Sp(2, \mathbb{Z})$ found by Seiberg and Witten[1] for the case $G = SU(2)$ arises from Picard-Fuchs equations satisfied by the holomorphic vector one-form $U_i = (\partial_i X^A, \partial_i F_A)$ $(i = 1, \dots, r)$ which can be regarded as differential identities for "rigid special geometry". For generic r, duality transformations act on the sections as[5]

$$\begin{pmatrix} \widetilde{X} \\ \widetilde{F} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X \\ F \end{pmatrix} \tag{2}$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2r, \mathbb{Z})$, leaving invariant the Kähler potential

$$K^{rigid} = -i(\overline{X}^A F_A - X^A \overline{F}_A) \ . \tag{3}$$

¿From the point of view of the $N = 2$ lagrangian, it can be shown that the theory is left invariant for $C = B = 0$, while for $B = 0$ one has perturbative and for $B \neq 0$ non perturbative dualities[5].

The above considerations brought immediately to mind the similar scenario encountered in the study of Calabi-Yau moduli space[23], where however the $N = 2$ supersymmetry was local[21].

¿From the point of view of the counting of degrees of freedom, in the coupling to gravity of the gauge theory on a group G broken to $U(1)^r$ for generic values of the moduli, there is always an additional $U(1)$ factor associated to the graviphoton field $G_{\mu i}$, corresponding to a symplectic section X^0, which is responsible for a drastic change in the geometry of moduli space. The main difference is that, in the local case, the moduli space is a Kähler–Hodge manifold rather than simply Kähler (that is, it possesses an extra $U(1)$ connection), with Kähler potential given by

$$K^{local} = -log\ i(\overline{X}^\Lambda F_\Lambda - X^\Lambda \overline{F}_\Lambda) \quad , \quad \Lambda = 0, 1, \dots, r \tag{4}$$

and three-form cohomology substitutes one-form cohomology[5]. Moreover, when dealing with those gauge theories that come as low energy effective field theories of strings one must also consider the extra $U(1)$ factor coming from $B_{\mu i}$, the vector partner of the dilaton-axion multiplet.

However, *mutatis mutandis*, it was quite natural to suppose that in the gravity coupled theory, the special geometry of the moduli space of an appropriate Calabi-Yau threefold with hodge number $h_{21} = r + 1$ should replace the rigid special geometry of the moduli space of the hyperelliptic Riemann surfaces Σ_r[4,5,16].

For a Calabi–Yau manifold defined as the vanishing locus of a certain algebraic surface $\mathcal{W}(\vec{x}; u_i)$ in a projective space, duality and monodromy groups are related quite generally by the relation[24]

$$\Gamma_D = \Gamma_M \ltimes \Gamma_{\mathcal{W}} \tag{5}$$

that is, the duality group is given by the semi-direct product of the monodromy group and the group $\Gamma_{\mathcal{W}}$ which contains the symmetries of $\mathcal{W}$. These consist of those linear transformations $\vec{x} \to M\vec{x}$ of the quasi-homogeneous coordinates $\vec{x}$ such that

$$\mathcal{W}(M\vec{x}; u_i) = f(u)\mathcal{W}(\vec{x}; \phi(u)) \tag{6}$$

where $\phi(u)$ is a generally non-linear transformation of the moduli u_i and $f(u)$ is a compensating overall rescaling of $\mathcal{W}$. Thus, it is clear that finding the monodromy group does not give the whole story. However, an important fact is that, due to the different form of the Kähler potential for local and rigid theories, in the latters only those trasformations of the defining polynomial of Σ_r corresponding to unimodular rescaling factors, rather than the full Γ_{Σ_r}, are true symmetries of the theory [25,16] (they are actually interpreted as R-symmetries[26]).

One expects that the right Calabi-Yau describing the moduli space of the gravitationally coupled theories must embed in some way the surfaces Σ_r of the rigid case, their monodromy and R-symmetry group[16].

Below, we focus on the rigid theories and and on the specific problem of determining Γ_D, and we briefly display an efficient method for the construction of the monodromy group $\Gamma_M(r) \in Sp(2r; \mathbb{Z})$ for the subclass Σ_r of hyperelliptic surfaces for the $SU(r+1)$ gauge theory. Explicitly, they have been shown to be given by[2,3]

$$w^2 = \left[z^{r+1} - \sum_{i=1}^{r} u_i \; z^{r-i} \right]^2 - \Lambda^{2r+2} \; . \tag{7}$$

where the coefficients u_i are the moduli and Λ is the dynamically generated scale. In our approach, the required monodromy group is selected as a particular subgroup of the monodromy group of the most generic hyperelliptic surface $\mathcal{W}_r$ of genus r

$$w^2 = P_{(2r+2)}(z) = z^{2r+2} + c_1 \; z^{2r+1} + \cdots + c_{2r+1} \; z + c_{2r+2} = \Pi_{i=1}^{2r+2}(z - \lambda_i) \; , \tag{8}$$

where only $2r-1$ of the c_k are independent moduli. The method presented here yields a complete solution for any $SU(r+1)$ and is based on some tools that were introduced in [27]. Our results can be compared with those recently obtained in[28], where the particular case of $SU(3)$ has been thoroughly discussed and where the corresponding periods of the theory have been obtained. The monodromy for $SO(2r+1)$ gauge theories was studied in [29].

The basic observation is that the monodromy group for $\mathcal{W}_r$ is given by a $2r$-dimensional representation of $B(2r+2)$, the braid group acting on $2r+2$ strands, on the homology basis of $\mathcal{W}_r$. Indeed, the monodromy group of a p-fold $\mathcal{M}$ is given by the representation on the homology basis of the p-fold of the fundamental group π_1 of the complement of the bifurcation set of $\mathcal{M}$. For the case $\mathcal{M} = \mathcal{W}_r$, where

$\mathcal{W}_r$ is described by the polynomial in Eq. (7), denoting by $Q^{(2r-2)}$ the bifurcation set of Eq. (7), and by C the base point, we have

$$\pi_1(CP^{(2r-1)} - Q^{(2r-2)}; C) \equiv B(2r+2) \ , \tag{9}$$

since $B(2r+2)$ is the fundamental group of the space of polynomials of degree $2r+2$ with no multiple roots. The bifurcation set of a polynomial is given by the submanifold in the moduli space $\{c_1, \ldots, c_{2r-1}\}$ where two or more roots λ_i coincide.

The generators t_i of $B(2r+2)$ correspond to the exchange of the i-th and the $i+1$-th strand and satisfy the relations

$$\begin{aligned} t_i t_{i+i} t_i &= t_{i+1} t_i t_{i+1} \\ t_i t_j &= t_j t_i \qquad\qquad |i-j| \geq 2 \ . \end{aligned} \tag{10}$$

In particular, to each generator $t_i \in \pi_1 \equiv B(2r+2)$ there corresponds a loop in the moduli space which exchanges the roots λ_i, λ_{i+1} of the polynomial and a vanishing cycle of $\mathcal{W}_r$. For a generic hyperelliptic surface any two roots can be exchanged by a suitable word in the generators t_i.

Let us now consider the particular subclass of hyperelliptic surfaces $\Sigma_r \in \mathcal{W}_r$. Their peculiarity is that they can be written in a factorized form

$$P_{(2r+2)}(z) = P^2_{(r+1)}(z) - P^2_{(1)} = \left[P_{(r+1)}(z) + P_{(1)}(z)\right] \left[P_{(r+1)}(z) - P_{(1)}(z)\right] \ , \tag{11}$$

where $P_{(r+1)}(z)$ and $P_{(1)}$ are two polynomials respectively of degree $r+1$ and 1. Altogether they contain $r+3$ parameters that can be identified with the $r+1$ roots of $P_{(r+1)}$ and the two coefficients of $P_{(1)}$,

$$P_{(r+1)}(z) = \Pi_{i=1}^{r+1}(z-\lambda_i) \quad , \quad P_{(1)}(z) = \mu_1 z + \mu_0 \ . \tag{12}$$

Furthermore, three conditions can be imposed to fix fractional linear transformations,

$$\sum_{i=1}^{r+1} \lambda_i = 0 \quad , \quad \mu_1 = 0 \quad , \quad \mu_0 = \Lambda^{r+1} \tag{13}$$

so that to leave effectively only r parameters. Of course other gauge choices would be completely equivalent, and may actually be convenient[16]. Corresponding to the factorization in Eq. (11) we have a natural splitting of the $2r+2$ roots of $P_{(2r+2)}$ into two sets

$$\{\lambda_1, \ldots, \lambda_{r+1}\} \quad and \quad \{\lambda_{r+2}, \ldots, \lambda_{2r+2}\} \ . \tag{14}$$

It is obvious that for the particular surface Σ_r the fundamental group π_1 mentioned above will be generated by those elements $t_i \in B(2r+2)$ which respect the splitting in Eq. (14), that is

$$\{t_1, \ldots, t_r, t_{r+1}^2, t_{r+2}, \ldots, t_{2r+2}, T = (t_1 t_2 \cdots t_{2r+1})^{r+1}\} \tag{15}$$

where T corresponds to the exchange of the two sets of roots ($t_1 \cdots t_{2r+1}$ corresponds to the cyclic permutation $\{\lambda_1, \lambda_2, \ldots, \lambda_{2r+2}\} \to \{\lambda_2, \lambda_3, \ldots, \lambda_{2r+2}, \lambda_1\}$). We conclude that the fundamental group of the hyperelliptic surface Σ_r is generated by the elements in Eq. (15). The required monodromy group $\Gamma_M[r]$ is therefore given by the representation on the homology group $H_1(\Sigma_r, \mathbb{Z})$ of the generators (15). At this point the strategy for computing the explicit monodromy of Σ_r is clear: one first obtains the monodromy group of $\mathcal{W}_r$ as a representation $M(t_i)$ of the $B(2r+2)$ generators on the homology basis of Σ_r. Then the monodromy group of Σ_r is given by the subgroup generated by

$$\{M(t_1), \ldots, M(t_r), M^2(t_{r+1}), M(t_{r+2}), \ldots, M(t_{2r+1}), M(T)\} \tag{16}$$

Let us then construct the monodromy group of $\mathcal{W}_r$ as a representation of $B(2r+2)$ on $H_1(\mathcal{W}_r; \mathbb{Z})$. We first choose a basis of cycles (A^I, B_I) on the cut z-plane such that

$$A^I \cap A^J = B_I \cap B_J = 0, \;\; A^I \cap B_J = -B_J \cap A^I = \delta^I_J \quad (I, J = 1, \ldots, r) \tag{17}$$

so that the homology intersection form C takes the canonical form

$$C = \begin{pmatrix} 0 & \mathbb{1}_r \\ -\mathbb{1}_r & 0 \end{pmatrix} \tag{18}$$

Actually we may take the cycles A^I to encircle the couple of roots $(\lambda_{2I}, \lambda_{2I+1})$, while the B_I cycles encircle the set of roots $(\lambda_1, \lambda_2, \ldots, \lambda_{2I})$. To a generic element $t \in B(2r+2)$ we may associate the corresponding vanishing cycle L of $H_1(\mathcal{W}_r, \mathbb{Z})$, say

$$L = (n^e_I, n^I_m) \tag{19}$$

where (n^e_I, n^I_m) are the components of L with respect to the basis (A^I, B_I). Using the Picard–Lefschetz formula[28]

$$\delta \to \delta - (\delta \cup L)L \tag{20}$$

which represents the transformation induced on the homology by the vanishing cycle L corresponding to the element $t \in B(2r+2)$, it is easy to see that in the given basis the corresponding monodromy matrix $M(L)$ is given by:

$$M(L) = \mathbb{1} + L \otimes (CL) \equiv \begin{pmatrix} \mathbb{1} + \vec{n}_e \otimes \vec{n}_m & -\vec{n}_e \otimes \vec{n}_e \\ \vec{n}_m \otimes \vec{n}_m & \mathbb{1} - \vec{n}_m \otimes \vec{n}_e \end{pmatrix} . \tag{21}$$

Denoting by $L^{(i)}$ the homology element associated to t_i ($i = 1, \ldots, 2r+2$), their explicit form is found by imposing the braid relations (9) on $M(L^{(i)})$, which yield the constraints

$$\begin{aligned} L^{(i)T} C L^{(j)} &= 0 \qquad\qquad |i-j| \geq 2 \\ L^{(i)T} C L^{(i+1)} &= 1 \ . \end{aligned} \tag{22}$$

The solution can be written as follows

$$\begin{aligned} L^{(2j-1)} &= (\vec{e}_j - \vec{e}_{j-1}; \vec{0}) \\ L^{(2j)} &= (\vec{0}; -\vec{e}_j) \qquad\qquad j = 1, \ldots, r \\ L^{(2r+1)} &= (-\vec{e}_r; \vec{0}) \\ L^{(2r+2)} &= (\vec{0}; \vec{e}_1 + \cdots + \vec{e}_r) \ , \end{aligned} \tag{23}$$

where $\vec{e}_i$ is an orthonormal basis in $\mathbf{R}^r$ ($\vec{e}_0 = 0$). Notice that the electric charges of the odd-numbered L and the magnetic charges of the even-numbered L are given in terms of the roots and fundamental weights of $SU(r+1)$ [28]. The restriction of the braid group generated by $M(L^{(i)})$ to the subgroup given in Eq. (16) (with $(M(L^{(i)}) \equiv M(t_i))$ gives the monodromy group of the hyperelliptic curves Σ_r .

We stress that our construction selects uniquely the possible entries of $L^{(i)} = (\vec{n}_e^{(i)}, \vec{n}_m^{(i)})$, corresponding to the values of the allowed electric and magnetic charges of any $SU(r+1)$ gauge theory.

Finally, we comment on the R-symmetry group pertaining to the rigid theory, that is on the $\Gamma_{\mathcal{W}}$ part of the duality group[16]. The defining polynomial of a generic hyperelliptic surface is known to admit a symmetry group that is isomorphic to the dihedral group D_{2r+2}, defining by the following relations on two generators A, B

$$A^{2r+2} = \mathbb{1} \ , \quad B^2 = \mathbb{1} \ , \quad (AB)^2 = \mathbb{1} \ . \tag{24}$$

However, for the specific subclass Σ_r, all that survives is the cyclic subgroup $\mathbb{Z}_{2r+2} \in D_{2r+2}$ generated by A. These symmetries, together with Γ_M found above, yield the full electromagnetic duality group.

We note that all we needed to apply this method was the defining equation of the relevant algebraic curve, and thus it is suitable to be applied to other cases in which such curve can be ultimately expressed as a polynomial in a single variable.

The restrictions found on the possible values of the electric and magnetic quantum numbers could be used in the general expression for the central charge for BPS states[1]

$$\mid Z \mid = \mid n_A^{(e)} X^A - n^A_{(m)} F_A \mid \tag{25}$$

to determine the exact spectrum of the theory.

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