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Four-Dimensional Avatars of Two-Dimensional RCFT

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We investigate a 4D analog of 2D WZW theory. The theory turns out to have surprising finiteness properties and an infinite-dimensional current algebra symmetry. Some correlation functions are determined by this symmetry. One way to define the theory systematically proceeds by the quantization of moduli spaces of holomorphic vector bundles over algebraic surfaces. We outline how one can define vertex operators in the theory. Finally, we define four-dimensional "conformal blocks" and present an analog of the Verlinde formula.

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1. Introduction and Conclusion

Two-dimensional rational conformal field theories are completely solvable and have formed the basis for much progress in understanding quantum field theory. It is therefore natural to ask if there exist 4D QFT's which are as "solvable" as 2D rational conformal field theories. The aim of the present paper is to outline just such a class of theories, we will refer to them as 4D WZW theories, or WZW_4, for short. These theories have infinite dimensional symmetry algebras generalizing those of the two-dimensional WZW model. Our results indicate that a 4D analog of RCFT probably exists. Much work remains to be done to flesh out this outline.

In constructing WZW_4 one learns an important philosophical lesson. In cohomological field theories [1][2][3] a *topological sector* is embedded in a more complicated nontopological theory. In the WZW_4 theory close relations to algebraic geometry rapidly become apparent, and it appears that there is an intermediate sector of the theory which is exactly solvable using methods of algebraic geometry. We refer to this sector as the *algebraic sector* of the theory. We postulate that there is a class of *algebro-geometric quantum field theories* analogous to the class of topological field theories, in the sense mentioned above.

In 2D RCFT the first order "bc systems" [4] play an important role. An interesting four-dimensional generalization of these theories exists and, at least in the algebraic sector, provides a four-dimensional analogue of nonabelian bosonization [5].

This note is a much-truncated version of a longer manuscript [5], which contains fuller explanations, more precise statements, and some results on representation theory, fermionization, and symplectic volumes not covered here. One potentially important application of the present work is to nonperturbative string theory, since the theory we discuss is related to the string field theory of the $N = 2$ string [6][7].

The results in section 5.3 were obtained in collaboration with Ian Grojnowski and also appear in [8].

2. WZW_4: Definition and Properties of Classical Theory

2.1. Lagrangian.

Let X_4 be a four-manifold equipped with a metric $h_{\mu\nu}$ and a closed 2-form $\omega \in \Omega^2(X;\mathbb{R})$. Let $g \in Map(X_4, G)$ for a Lie group G. Fix a reference field configuration $g_0(x)$. Then, for any $g(x)$ in the same homotopy class as $g_0(x)$ we may define a natural analog of the D=2 WZW theory by the Lagrangian:

$$S_\omega[g;g_0] = \frac{f_\pi^2}{8\pi}\int_{X_4} \mathrm{Tr} g^{-1}dg \wedge *(g^{-1}dg) + \frac{i}{12\pi}\int_{X_5} \omega \wedge \mathrm{Tr}(g^{-1}dg)^3 \tag{2.1}$$

Here f_π is a dimensionfull parameter. [1] In the integral over $X_5 = X_4 \times I$ in (2.1) ω is independent of the fifth coordinate; moreover, we use a homotopy of g to g_0. The action is independent of the choice of homotopy up to a multiple of the periods of ω. [2]

2.2. Classical Equations of Motion. Kähler Point

The classical equations of motion, following from (2.1), are:

$$\begin{aligned} &d * g^{-1}dg + \omega \wedge g^{-1}dg \wedge g^{-1}dg = \\ &\partial_\mu h^{\mu\nu}\sqrt{h} g^{-1}\partial_\nu g + \epsilon^{\alpha\beta\gamma\delta}\omega_{\alpha\beta} g^{-1}\partial_\gamma g g^{-1}\partial_\delta g = 0 \end{aligned} \tag{2.2}$$

They simplify drastically in the case where X_4 is a complex four-manifold with Kähler metric with ω the associated Kähler form:

$$\omega = \frac{i}{2} f_\pi^2 h_{i\bar{j}} dz^i \wedge dz^{\bar{j}} \tag{2.3}$$

We refer to this point in the space of Lagrangians as the "Kähler point."

Using standard properties of the Hodge star we may rewrite the action as:

$$S_\omega[g] = -\frac{i}{4\pi}\int_{X_4} \omega \wedge \mathrm{Tr}\big(g^{-1}\partial g \wedge g^{-1}\bar{\partial} g\big) + \frac{i}{12\pi}\int_{X_5} \omega \wedge \mathrm{Tr}(g^{-1}dg)^3 \tag{2.4}$$

The equations of motion following from (2.4) are:

$$\omega \wedge \bar{\partial}\big(g^{-1}\partial g\big) = 0 \tag{2.5}$$

These equations are known as the Yang equations, and are equivalent to the self-dual Yang-Mills equations.

Remarks .

[1] Our conventions are the following: Differential forms are considered dimensionless, but dx^μ carries dimension one. Thus, the metric $g_{\mu\nu}$ is dimensionless, but f_π^2, $\omega_{\mu\nu}$ are dimension -2, etc.

[2] Note that X_5 is a cylinder, rather than a cone. This is necessary since X_4 might not be cobordant to zero, and since the periods of ω might be nontrivial. As was noted in [9], the latter fact caused difficulties in finding a "Mickelsson"-type construction [10] of $\widehat{\mathrm{Map}}(\Sigma, G)$ where Σ is a Riemann surface. Using a cylinder and a homotopy construction this problem can be overcome [11]. A completely different solution to this problem has recently been described in [12].

1. The Lagrangian (2.4) was first written by Donaldson [13]. It was studied by Nair and Schiff as a natural generalization of the 2D CFT/ 3D CSW correspondence [14][15].
2. In [6][7] the special Kähler point is related to the classical field theory of the $N = 2$ string. String investigations of this theory have focused on the S-matrix for π defined by $g = e^{\pi}$. The present paper studies field configurations g related to instantons. Hence one may expect that results of this paper will have a bearing on nonperturbative $N = 2$ string theory.

2.3. Twistor transform and Classical Integrability

We explain how one can solve equations of motion in WZW_4 theory using the twistor transform of the self-dual Yang-Mills equations.

$\mathbb{R}^4$ can be endowed with complex structures parametrized by $\mathbb{P}^1$. Choosing a base-point complex structure z^1, z^2 the others are defined by: $z_u^A = z^A + u\epsilon^{AB}\bar{z}^B$ and u labels a point $u \in \hat{\mathbb{C}} \cong \mathbb{P}^1 \cong SO(4)/U(2)$.

Given a region $\mathcal{R} \subset \mathbb{R}^4$ its twistor space $\hat{\mathcal{R}} \to \mathcal{R}$ has as fiber the sphere $\mathbb{P}^1$ of complex structures compatible with an orientation. The twistor transform defines a correspondence between solutions of the Yang equation $\mathcal{Y}(\mathcal{R})$ and twistor data $\mathcal{T}(\hat{\mathcal{R}})$ defined by:

$$\begin{aligned} \mathcal{Y}(\mathcal{R}) &\equiv \{g : \omega \wedge \bar{\partial}(g^{-1}\partial g) = 0 \quad \text{on} \quad \mathcal{R}\} \\ \mathcal{T}(\hat{\mathcal{R}}) &\equiv \{G(s^1, s^2, u) : \hat{\mathcal{R}}_+ \cap \hat{\mathcal{R}}_- \to GL(n, \mathbb{C})\} \end{aligned} \tag{2.6}$$

where $\hat{\mathcal{R}}_\pm$ are patches defined by the north/south pole and:

1. $G(z_u^1, z_u^2, u) = H_-^{-1}(x, u)H^+(x, u)$ for $0 < |u| < \infty$.
2. $H_\pm(x, u)$ holomorphic in u for $\mid u \mid < \infty, \mid u \mid > 0$

Briefly, choosing a gauge $A^{(1,0)} = 0, A^{(0,1)} = -\bar{\partial}gg^{-1}$ the SDYM are satisfied iff $\forall u, F^{(0,2),u} = 0$ which holds iff $\bar{\partial}_{\bar{A}}^{(0,1),u} H(x, u) = 0$. Imposing holomorphy of $H(x, u)$ in u forces us to choose two functions $H_\pm$ holomorphic on the patches $\hat{\mathcal{R}}_\pm$ and related as in item 1 above. We then identify $g = H_+(x, u = 0)$ as a solution to the Yang equation (this construction is analagous to the one in [16]).

Note that all solutions to the Yang equation could be obtained by taking different holomorphic functions G and solving the Riemann-Hilbert problem (1.) In this sense the classical equations of motion for a complex group G are integrable.

It is also worth noting that the twistor transform gives a generalization to 4D of the important property of holomorphic factorization in WZW_2 [5].

2.4. Canonical Approach to the Classical Theory and Current Algebra

We consider a four-manifold with space-time splitting $X_4 = X_3 \times \mathbb{R}$. [3] The phase space of the model can be identified with $\mathcal{P} = T^*Map(X_3, G)$, where the momenta are valued in: $I^a(x) \in \Omega^3(X_3, \underline{\mathbf{g}})$ and $\underline{\mathbf{g}}$ is the Lie algebra of G.

Writing the action S_ω in first order form we extract the symplectic form:

$$\Omega_\omega = \int_{X_3} \mathrm{Tr}\left[\delta I \wedge g^{-1}\delta g - \left(I + \frac{1}{4\pi}\omega \wedge g^{-1}dg\right)(g^{-1}\delta g)^2\right] \tag{2.7}$$

from which we obtain the commutation relations of $I^a(x), g(x)$.

$$\begin{aligned} [I^a(x), I^b(y)]_\omega &= f_c^{ab}(I + \frac{1}{4\pi}\omega \wedge g^{-1}dg)^c \delta^{(3)}(x-y) \\ [I^a(x), g(y)]_\omega &= g(y)T^a\delta^{(3)}(x-y) \qquad . \end{aligned} \tag{2.8}$$

From these relations we can obtain the generalization of two-dimensional current algebra. Form the combination

$$J^a(x) = I^a(x) - \frac{1}{4\pi}\omega \wedge (g^{-1}dg)^a(x) \tag{2.9}$$

For a $\underline{\mathbf{g}}$-valued function on X_3 $\epsilon^a(x)$ it gives rise to the charge:

$$Q(\epsilon) = \int_{X_3} \mathrm{Tr}(\epsilon J) \tag{2.10}$$

The charges $Q(\epsilon)$ obey the following commutation relations:

$$\{Q(\epsilon_1), Q(\epsilon_2)\} = Q([\epsilon_1, \epsilon_2]) + \int_{X_3} \omega \wedge \mathrm{Tr}(\epsilon_1 d\epsilon_2) \tag{2.11}$$

We denote this algebra as $\kappa(X_3, \underline{\mathbf{g}}, \omega)$.

Remarks.

1. There are several possible generalizations of 2D current algebra. The above algebra is the one relevant to the *algebraic* sector of our theory, but, for example, using the commutation relations (2.8) it is possible to form a larger algebra as follows. We can form the objects like $I_\xi = I - \frac{1}{4\pi}(\omega + \xi) \wedge g^{-1}dg$, where ξ stands for *any* two-form on X_3. These form an algebra of charges $Q_\xi(f) = \int_{X_3} \mathrm{Tr}(fI_\xi)$:

$$\begin{aligned} \{Q_{\xi_1}(f_1), Q_{\xi_2}(f_2)\} &= Q_{\xi_1+\xi_2}([f_1, f_2]) + \\ &+ \int_{X_3}\left[(\omega + \xi_1)\mathrm{Tr}(f_1 df_2) - (\omega + \xi_2)\mathrm{Tr}(f_2 df_1)\right] \end{aligned} \tag{2.12}$$

[3] We continue to work with Euclidean signature

2. There is an interesting similarity between the algebra $\kappa(X_3, \underline{\mathbf{g}}, \omega)$ and the algebra discovered in [17][18][19][20] for the case when an abelian gauge field strength is equal to a Kähler form ω. This analogy suggests the existence of a free field interpretation of the algebraic sector (see below) of WZW_4. Indeed, such an interpretation of the algebraic subsector of the theory exists and will be described in [5].

3. WZW_4: Quantum theory

3.1. Quantization of $[\omega]$ and the first sign of algebraic geometry

As usual, the coefficient of the WZ term is quantized. Two different homotopies of g to g_0 define a map $g : S^1 \times X \to G$. Consequently, if the group G is nonabelian, the measure $\exp iS$ in the path integral is only well-defined if

$$\omega \wedge \frac{1}{12\pi}\mathrm{Tr}(g^{-1}dg)^3 \in H^5(S^1 \times X_4; 2\pi\mathbb{Z}) \tag{3.1}$$

which forces the cohomology class $[\omega]$ to lie in the lattice:

$$[\omega] \in H^2(X_4; \mathbb{Z}) \tag{3.2}$$

The class $[\omega]$ is the four-dimensional analog of k. Note that although $[\omega]$ is quantized the Lagrangian depends on the representative of the class. Since ω is of type $(1,1)$ condition (3.2) implies $[\omega] \in H^2(X;\mathbb{Z}) \cap H^{1,1}(X;\mathbb{R})$ so the metric is Hodge, and, by the Kodaira embedding theorem, if X_4 is compact, it must be algebraic [21]. [4]

3.2. Remarks on States and Hilbert Spaces: An operator formalism in 4d

One goal of this investigation is the generalization of concepts of 2D CFT to 4D. In the 2D case one fruitful approach to defining states and Hilbert spaces proceeds by considering states defined by path integrals on 2-folds with boundary, the boundary is interpreted as a spatial slice.

In 2D a compact connected spatial slice is necessarily a copy of S^1. In 4D, on the other hand, there is a wide variety of possible notions of space. Two natural choices are S^3, corresponding to radial propagation around a point and a circle bundle over an embedded surface, corresponding to radial propagation in the normal bundle to the surface.

[4] Curiously, the supersymmetric σ model can only be coupled to $N = 1$ supergravity when the target is a Hodge manifold [22], for similar reasons.

Suppose now that the 3-fold bounds a 4-fold $Y = \partial X$, then a particular state $\Psi_Y \in \mathcal{H}(Y)$ is defined by the path integral. An important problem is to describe how one can construct this state explicitly. One immediately encounters a fundamental difficulty in defining the state via conserved charges called the "2/3 problem."

Conserved charges in 2D come from (anti-) holomorphic currents. If $f|_{X_1}$ extends to a holomorphic function on X_2 then by contour deformation:

$$\bar{\partial}(g^{-1}\partial g) = 0 \quad \Rightarrow \quad \oint_{X_1} \mathrm{Tr}(f g^{-1}\partial g)\Psi = 0 = \oint_{X_1} \mathrm{Tr}(\bar{f}\bar{\partial} g g^{-1})\Psi \tag{3.3}$$

In 2D, "enough" boundary values f extend to holomorphic or anti-holomorphic functions on X_2 that the conserved charges determine the state up to a finite-dimensional vector space (related to conformal blocks).

In 4D, $\omega \wedge g^{-1}\partial g$ is a $\bar{\partial}$-closed $(2,1)$ form, and hence, if $f|_{X_3}$ extends to a holomorphic function on X_4 "contour deformation" of X_3 implies

$$\oint_{X_3} \mathrm{Tr}\left[f(z^1, z^2)\omega \wedge g^{-1}\partial g \right] \Psi = 0 \tag{3.4}$$

but note there are not nearly enough such functions to determine the vacuum. Even in the best cases [5] we are missing one *functional* degree of freedom: boundary values of holomorphic functions depend on two functional degrees of freedom but to determine the state we need conserved charges depending on three functional degrees of freedom.

One possible approach to the problem uses the twistor transform of the SDYM equations. In the abelian case it does solve the problem of missing charges - harmonic functions on a ball with the flat metric are linear combinations of holomorphic functions in different complex structures.

3.3. The perturbative vacuum is one-loop finite

The WZW_4 theory possesses unusual finiteness properties. The one-loop renormalization of the vacuum state may be carried out via the background-field method. We let $g = e^{i\pi/f_\pi} g_{cl}$ where g_{cl} is a solution of the classical equations of motion with given boundary conditions. For one-loop renormalization it suffices to keep the terms quadratic in $\pi(x)$:

$$S = S_{cl} + \int (\nabla_\mu (J_{cl})\pi)^2 + \pi^a M^{ab} \pi^b + \mathcal{O}(\pi^3) \tag{3.5}$$

[5] for example, in radial quantization around a point

where the connection $\nabla_\mu(J_{cl})$ and M^{ab} are constructed from the classical solution of the equation of motion. The divergent terms at one-loop may be extracted from the Seeley expansion:

$$\Delta S = \int \frac{1}{\epsilon^4} C + \frac{1}{\epsilon^2} \mathrm{Tr} M + (\log \epsilon) \mathrm{Tr} \left[\frac{1}{2} M^2 + \frac{1}{6} (\nabla^2 M - \frac{1}{2} F_{\mu\nu}(J)^2) \right] + \cdots \tag{3.6}$$

where C is a π- independent constant and $F_{\mu\nu}(J)$ is a fieldstrength constructed from $\nabla_\mu(J_{cl})$. For the WZW_4 action we find that

$$M = 0, \quad \bar{\nabla}(J_{cl}) = \bar{\partial}, \quad \nabla(J_{cl}) = \partial + [g_{cl}^{-1} \partial g_{cl}, \cdot].$$

We conclude that there is no quadratic divergence at the Kähler point and in addition, using the equations of motion we see that the logarithmic divergences also vanish once boundary terms are properly included. Of course, these statements can also be checked directly using Feynman diagrams.

3.4. Discussion of Renormalizability

Given the surprising one-loop finiteness one may naturally wonder if the model is finite. This remains a mystery. Several arguments indicate that the renormalizability properties of the theory are special, but eventually they remain inconclusive.

Even if the theory is 2-loop infinite we should not give up. WZW_4 is clearly a very distinguished theory. In particular, as we shall see, the relation to the moduli of instantons is completely parallel to that of WZW_2 to the moduli space of flat connections, and one can elevate this statement to a defining principle of the quantum theory. Thus, even if it cannot be defined by standard field-theoretic techniques, we believe the theory exists. This, in fact, is one of the sources of its great interest. Therefore, in the sequel, we choose another way of defining the correlation functions.

3.5. The PW formula and the quantum equations of motion

At the "Kähler point" the classical equations of motion have a local "two-loop group" symmetry $\mathcal{H}G_{\mathbb{C}} \times \mathcal{H}G_{\mathbb{C}}$, where $\mathcal{H}G_{\mathbb{C}} = \{g(z^1, z^2) \in G_{\mathbb{C}}\}$, which takes $g \to g_L(\bar{z}^i) g(z, \bar{z}) g_R(z^i)$ As in 2D this is related to the Polyakov-Wiegman (PW) formula [23]:

$$S_\omega[gh] = S_\omega[g] + S_\omega[h] - \frac{i}{2\pi} \int_{X_4} \omega \wedge \mathrm{Tr}[g^{-1} \partial g \bar{\partial} h h^{-1}] \tag{3.7}$$

Assuming invariance of the path integral measure Dg with respect to the left action of the group $\mathrm{Map}(X, G)$ (with G compact) we can derive the quantum equations of motion:

$$\begin{aligned} \bar{\partial} \langle J \rangle &= 0 \\ J &= \omega \wedge g^{-1} \partial g \end{aligned} \tag{3.8}$$

3.6. Ward identities

The identity (3.7) implies that the $(2,1)$-form current $J = \omega \wedge g^{-1}\partial g$ satisfies:

$$\begin{aligned}\frac{i}{2\pi}\bar{\partial}_x \left\langle J^A(x) \prod J^{A_i}(x_i) \right\rangle = \sum_i (\omega \wedge \partial)_i (\delta(x,x_i)) \delta^{A,A_i} \left\langle \prod_{j\neq i} J^{A_j}(x_j) \right\rangle \\ + \sum_i f^{AA_iB_i}(\delta(x,x_i)) \left\langle J^{B_i}(x_i) \prod_{j\neq i} J^{A_j}(x_j) \right\rangle\end{aligned} \tag{3.9}$$

where $\delta(x,y)$ is a 4-form in x and a 0-form in y.

In strong contrast to the situation in 2D, the identities (3.9) do *not* determine the correlation functions since $\bar{\partial}$ has an ∞-dimensional kernel and these identities cannot fully determine the correlation functions. However, the identities, together with simple analyticity arguments do determine many correlation functions in a way analogous to the 2D case. Indeed, suppose Y is a compact three-manifold, which divides X_4 in two parts: X_+ and X_-, and suppose that f is a function on Y taking values in the Lie algebra $\underline{\mathbf{g}}$ and extending holomorphically to X_+. Then the integral

$$V(f,Y) = \int_Y \mathrm{Tr}(fJ), \tag{3.10}$$

generates an infinitesimal gauge transformation. The field g in the region X_+ gets transformed to $g + fg$, and remains unchanged in X_-. Now, keeping in mind the fact, that J transforms as

$$J \to J + \omega \wedge \partial f + [J, f], \tag{3.11}$$

we can derive the correlation functions of the currents, integrated with appropriate functions f. Examples will be presented in the next section.

3.7. Definition of $\mathcal{Z}(\bar{A})$ and its equivariance properties

It is natural to consider the generating function for the correlators of the currents J:

$$\mathcal{Z}(\bar{A}) = \int Dg e^{-S_\omega(g)} e^{-\frac{i}{2\pi}\int_{X_4} \mathrm{Tr} J\bar{A}} \tag{3.12}$$

The generating parameter $\bar{A}$ is a $\underline{\mathbf{g}} \otimes \mathbb{C}$-valued $(0,1)$ form on X_4. [6]

[6] $\bar{A}$ is actually a connection, but we will work on a topologically trivial bundle until section 4.4 below.

Leaving aside the question of the regularization of the quantity on the right hand side of (3.12) we observe the following equivariance property of $\mathcal{Z}$, following from the PW formula:

$$\mathcal{Z}(\bar{A}^{h^{-1}}) = e^{S_\omega(h)+\frac{i}{4\pi}\int_{X_4}\omega\wedge \mathrm{Tr}\, h^{-1}\partial h\bar{A}}\mathcal{Z}(\bar{A}) \tag{3.13}$$

Taking h to be infinitesimally close to 1 we get the following functional equation on $\mathcal{Z}$:

$$\widehat{F}^{(1,1)+}\mathcal{Z} = \omega\wedge F(\Lambda\frac{\delta}{\delta\bar{A}},\bar{A})\mathcal{Z} = 0 \tag{3.14}$$

where

$$F(A,\bar{A}) = \partial\bar{A} - \bar{\partial}A + [A,\bar{A}]$$

and Λ is an operation inverse to multiplication by ω. Namely, the action of Λ on a three form $\Omega_{\mu\nu\lambda}$ gives a one form

$$(\Lambda\Omega)_\mu = \frac{1}{2}\omega^{\nu\lambda}\Omega_{\mu\nu\lambda},$$

where the bivector $\omega^{\nu\lambda}$ is inverse to $\omega_{\mu\lambda}$. Moreover, since J satisfies the flatness condition:

$$\partial(\Lambda J) + (\Lambda J)^2 = 0, \tag{3.15}$$

$\mathcal{Z}$ obeys

$$\widehat{F}^{2,0}\mathcal{Z} = F^{2,0}(\Lambda\frac{\delta}{\delta\bar{A}})\mathcal{Z} = 0 \tag{3.16}$$

4. Algebraic Sector

In this section we assume that the quantum theory can be defined preserving invariance of the measure Dg.

4.1. Definition of the algebraic sector

From the PW identity we get

$$\left\langle \exp\left[-\frac{i}{2\pi}\int_{X_4}\omega\wedge \mathrm{Tr}[g^{-1}\partial g\bar{\partial}hh^{-1}\right]\right\rangle = \exp\left[S_\omega[h]\right] \tag{4.1}$$

This identity is closely related to the algebraic geometry of holomorphic vector bundles on X_4 and characterizes the content of the algebraic sector of the theory. In the following subsections we will extract some interesting information from this general statement.

Recall that $\mathcal{Z}(\bar{A})$ satisfies the two functional equations (3.14) and (3.16). Unfortunately, in the non-abelian theory the equation $\widehat{F}^{2,0}\mathcal{Z} = 0$ is non-linear and is apparently insoluble. Therefore, we shall restrict $\mathcal{Z}$ to the space of connections $\bar{A}$ such that $F^{0,2}(\bar{A}) = 0$. (Such connections will be called Kähler gauge fields and the space of such connections is denoted as $\mathcal{A}^{1,1}$.)

Our definition of the algebraic sector of WZW_4 theory is as follows: it is the set of correlators, which can be extracted from the properties of $\mathcal{Z}(\bar{A})$, evaluated on $\mathcal{A}^{1,1}$.

Naively, one can argue that the solution of the equation $\widehat{F}^{2,0}\mathcal{Z} = 0$ is uniquely determined by the restriction of $\mathcal{Z}$ onto the subspace $\mathcal{A}^{1,1}$, and thus all current correlators are determined by the algebraic sector.

4.2. Simplest examples of algebraic correlators: "Divisor current" correlators

In WZW_2 we study correlators of currents at points $z_1, \ldots, z_n$, i.e.

$$\langle j(z_1) \ldots j(z_n) \rangle.$$

Note that points are divisors in one dimensional complex geometry. For example, on $\mathbb{CP}^1$ a point z_1 is the polar divisor of the function $f_1(z) = 1/(z - z_1)$, and thus,

$$j(z_1) = \int_{Y_1} j(z) f_1(z). \tag{4.2}$$

where Y_1 is a small circle around the point z_1. As in (3.10)(3.11) this observable generates gauge transformations and hence its correlators are easily computed on $\mathbb{CP}^1$. Equivalently, the generating function of current correlators is a partition function in the presence of a field $\bar{A}$. This field defines the structure of a holomorphic vector bundle on an algebraic curve. If the bundle is holomorphically trivial we may find the correlators from the PW formula. Even if the bundle is nontrivial, but the curve is $\mathbb{CP}^1$ we may use singular gauge transformations (discussed below) to find the correlators. If the genus of the curve is greater than zero, the PW formula reduces the problem of the computation of current correlators to the problem of the quantization of the moduli space of holomorphic vector bundles.

In complex dimension two, arbitrary correlators of currents are generated by the function $\mathcal{Z}(\bar{A})$ for unrestricted $\bar{A}$, and hence cannot be computed within the algebraic sector. Nevertheless, it is possible to use the current to define an observable at a divisor D by analogy with the previous case, i.e. consider a meromorphic function $f \in \underline{\mathbf{g}} \otimes \mathbb{C}$ on

X_4 such that its polar divisor is D. Consider the three-dimensional boundary Y of a small tubular neighborhood around the divisor. Let us define an observable $J(f,D)$, associated to (f,D) as

$$J(f,D) = \int_Y \mathrm{Tr}(fJ) \tag{4.3}$$

This is just (3.10) above, and only depends on f, D.

Correlators of the observables (4.3) are generated by connections $\bar{A} \in \mathcal{A}^{1,1}$, and are therefore in the algebraic sector. Moreover, correlators of the observables (4.3) can be computed as follows. Suppose that the divisors D_i do not intersect one another so that their tubular neighborhoods do not intersect. Each observable generates holomorphic gauge transformations outside its tubular neighborhood and since the current transforms like a connection the "2-divisor" correlator is:

$$\Big\langle J(f_1,D_1)J(f_2,D_2)\Big\rangle = \int_{Y_2} \mathrm{Tr}\Big(f_2\omega \wedge \partial f_1\Big) \tag{4.4}$$

Similarly, the "3-divisor" correlator is:

$$\begin{aligned}\Big\langle J(f_1,D_1)J(f_2,D_2)J(f_3,D_3)\Big\rangle = &\int_{Y_2} \mathrm{Tr}\Big\{f_3\omega \wedge \partial\Big([f_1,f_2]\Big)\Big\} \\ &+ \int_{Y_3} \mathrm{Tr}\Big\{f_2\omega \wedge \partial\Big([f_3,f_1]\Big)\Big\}\end{aligned} \tag{4.5}$$

The formulae (4.4)(4.5) generalize current algebra on algebraic curves to current algebra on algebraic surfaces. Similarly the n-divisor correlators can be written as above.

Remark. One could also define observables analogous to (4.3) associated to arbitrary divisors, rather than polar divisors of meromorphic functions. These are still in the algebraic sector but their correlators cannot be computed as easily since they require information about nontrivial conformal blocks. [7]

4.3. Algebraic Correlators in the Abelian Theory

In some two-dimensional conformal field theories one can compute not only current correlators but also vertex operator correlators using singular gauge transformations. [8]

[7] Compare with the current correlator on a Riemann surface of genus > 0 in WZW_2.

[8] See, for example [24][25][26]. In [26] these relations were called "multiplicative Ward identities."

In the 4D $U(1)$ WZW_4 theory (which is just a Gaussian model) this method may be generalized as follows. [9]

We would like to make a singular complexified gauge transformation by a meromorphic function f:

$$g = e^{i\phi} \to fgf^\dagger \tag{4.6}$$

Intuitively, along the zero and polar divisors of f the effect of the singular gauge transformation is to insert a vertex operator. We can make this idea precise and compute the correlators using the (left- and right-) PW formula by using a regularized version of f. Suppose $div(f) = \sum n_i D_i$, and, near D_i, we may describe the divisor in local coordinates as $z_i = 0$. Thus we have

$$f = z_i^{n_i}(f_i + \mathcal{O}(z_i)) \tag{4.7}$$

where f_i can be meromorphic on D_i. We now use the Kähler metric to choose tubular neighborhoods T_i around D_i of radius ϵ_i, and use a partition of unity to define a regularized gauge transformation with parameter f_ϵ such that $|f_\epsilon|^2 = |f|^2$ outside $\cup T_i$, while $|f_\epsilon|^2 = (|z_i|^2 + \epsilon_i^2)^{n_i}|f_i + \cdots|^2$ in $T_i - \cup_{j\neq i} T_j$ etc.

Now consider the effect of such a gauge transformation, in the left-right version of the PW formula (4.1). The LHS of this formula involves a change of action by:

$$S_\omega \to S_\omega + \sum_i i n_i \int_{D^i} \omega\phi + \mathcal{O}(\epsilon) \tag{4.8}$$

as $\epsilon \to 0$. So the LHS formally becomes a correlator of operators

$$V_k[D] = e^{ik\int_D \omega\phi} \tag{4.9}$$

Computing the RHS of this formula we find, as $\epsilon \to 0$

$$S_\omega[\log|f_\epsilon|^2] \to \frac{1}{2}\sum_i \left[n_i^2 \log\epsilon_i^2 \int_{D_i} \omega + n_i \int_{D_i} \omega \log|f_i|^2\right] \tag{4.10}$$

[9] These remarks are closely related to ref. [27]. Another related example is given by Maxwell theory on X_4. Singular gauge transformations shift the line bundle and allow one to produce a classical partition function analogous to the partition functions of 2d gaussian models. They also allow one to study the vertex operators $\mathcal{V}(D) = \exp\left[\frac{i}{2\pi}\int_D F\right]$ and $\mathcal{V}^*(D) = \exp\left[\frac{i}{2\pi}\int_D *F\right]$. This and related issues have been independently studied in [28][29].

Exactly as in two dimensions the logarithmically divergent terms are renormalization factors needed to normal-order the observables (4.9). These observables accordingly have anomalous dimension

$$\Delta_k = \frac{k^2}{2}\int_D \omega \tag{4.11}$$

and the renormalized operators have the correlation function:

$$\left\langle \prod V_{n_i}[D_i] \right\rangle = \exp\left[\sum_i \frac{n_i}{2}\int_{D_i} \omega \log|f_i|^2\right] \tag{4.12}$$

Remark. We expect that the correlator will factorize holomorphically, as in two-dimensions, and that it should be possible to define "chiral vertex operators." One may expect a relation like:

$$\exp[\int_{D_0} ord_{D_0}(f)\omega\phi_L]\exp[\int_{D_\infty} ord_{D_\infty}(f)\omega\phi_L] \sim \exp\int_T \omega\wedge\partial\phi \tag{4.13}$$

where the zero and polar divisors form the boundary of a three-manifold T: $D_0 - D_\infty = \partial T$. The details of this proposal, especially in the nonabelian case, appear to be nontrivial and have not been worked out yet.

4.4. Moduli of Vector Bundles and ASD Connections

In two dimensions one of the possible ways of defining operators in the theory is through coupling the basic field g to non-trivial gauge fields. For example, in the abelian theory, by coupling to a line bundle $\mathcal{L}$ with non-trivial c_1 one can get insertions of operators at the points corresponding to the divisor of $\mathcal{L}$. This procedure can be generalized to the nonabelian case. Therefore, it is natural to try to generalize the WZW_4 action for the theory in the background of a non-trivial gauge field.

Let $E \to X_4$ be a rank r complex hermitian vector bundle with metric $(\cdot,\cdot)_E$ and Chern classes c_1, c_2 and let $g(x)$ be a section of $Aut(E)$. To write a well-defined action generalizing (2.4) we must introduce a connection ∇ on E. It gives rise to a connection on $Aut(E)$, which we also denote as ∇. Since X_4 is complex we can split the connection into its $(1,0)$ and $(0,1)$ pieces: $\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$. The WZW_4 action in the background (E,∇) is (we use here the result of [30]):

$$\begin{aligned} S_{\omega;E,\nabla}[g] = -\frac{i}{4\pi}\int_{X_4}\omega\wedge \mathrm{Tr}\big(g^{-1}\nabla^{(1,0)}g\wedge g^{-1}\nabla^{(0,1)}g\big) + \\ +\frac{i}{12\pi}\int_{X_5}\omega\wedge\left[\mathrm{Tr}(\tilde{g}^{-1}\tilde{\nabla}\tilde{g})^3 + 3\mathrm{Tr}\ F_{\tilde{\nabla}}[\tilde{g}^{-1}\tilde{\nabla}\tilde{g} + (\tilde{\nabla}\tilde{g})\tilde{g}^{-1}]\right)\quad . \end{aligned} \tag{4.14}$$

In the formula we trivially extended E to X_5 and we denote the extension of g, ∇ to X_5 as $\tilde{g}$, $\tilde{\nabla}$.

In order for the Polyakov-Wiegmann formula to be generalizable to non-trivial E we must check whether $S_{\omega;E,\nabla}[gh] - S_{\omega;E,\nabla}[g] - S_{\omega;E,\nabla}[h]$ is local in four dimensions. After a simple computation one obtains:

$$\begin{aligned} S_{\omega;E,\nabla}[gh] &= S_{\omega;E,\nabla}[g] + S_{\omega;E,\nabla}[h] \\ -\frac{i}{2\pi}\int_{X_4} \omega \wedge \mathrm{Tr}\big[g^{-1}(\nabla^{(1,0)}g)(\nabla^{(0,1)}h)h^{-1}\big] \end{aligned} \tag{4.15}$$

As before we define

$$\mathcal{A}^{1,1}(E \to X_4) \equiv \{\nabla : F^{0,2} = F^{2,0} = 0\} \tag{4.16}$$

to be the space of unitary connections whose curvature is of type $(1,1)$. If $\nabla \in \mathcal{A}^{1,1}$ then $\nabla^{(0,1)}$ is the Dolbeault operator defining an integrable holomorphic structure $\mathcal{E}$ on the vector bundle E. The moduli space of holomorphic vector bundles [10] is given by: $\mathcal{HB} \equiv \{\nabla^{(0,1)} = \nabla_0^{(0,1)} + \bar{A} : (\nabla^{(0,1)})^2 = 0\}/Aut(E)$, where $\nabla_0^{(0,1)}$ is some reference connection and $\bar{A}$ is a $(0,1)$-form with values in $ad(E)$.

We will perform a functional integral over the group of unitary automorphisms $Aut(E)^u$ of the bundle E . Once again we consider the generator of current correlation functions. For $\bar{A} \in \Omega^{0,1}(ad(E))$:

$$\mathcal{Z}_E[\bar{A}] = \left\langle \exp \frac{i}{2\pi}\int_{X_4} \omega \wedge \mathrm{Tr}\bar{A}g^{-1}\nabla_0^{(1,0)}g \right\rangle \tag{4.17}$$

By the generalized PW formula (4.15) we have the equivariance condition:

$$\mathcal{Z}_E(\bar{A}^{h^{-1}}) = e^{S_{\omega;E,\nabla_0}[h] + \frac{i}{2\pi}\int_{X_4}\omega\wedge\mathrm{Tr}\bar{A}h^{-1}\nabla_0^{(1,0)}h}\,\mathcal{Z}_E(\bar{A}) \tag{4.18}$$

where $\bar{A}^{h^{-1}} = h\bar{A}h^{-1} - (\nabla_0^{(0,1)}h)h^{-1}$. Restricting $\nabla_0^{(0,1)} + \bar{A}$ to be in $\mathcal{A}^{1,1}(E \to X_4)$ we learn that $\mathcal{Z}_E[\bar{A}] \in H^0(\mathcal{HB};\mathcal{L}_\omega)$ where the line bundle $\mathcal{L}_\omega$ has non-trivial first Chern class $c_1(\mathcal{L}_\omega) = \omega_{\mathcal{M}}$, associated with the Kähler form on $\mathcal{A}^{1,1}(E \to X_4)$; $\omega_{\mathcal{M}} = \int_X \omega_X \wedge \mathrm{Tr}\delta A \wedge \delta A$. In other words, $\mathcal{Z}_E[\bar{A}]$ is a wavefunction for the quantization of $\mathcal{HB}$.

The quantization can also be understood as the quantization of the moduli space $\mathcal{M}^+$ of ASD connections. Mathematically, this follows from the Donaldson-Uhlenbeck-Yau

[10] Technically one should specify carefully a compactification of this moduli space. One reasonable possibility is to choose the moduli space of rank r torsion free sheaves on X semistable with respect to a polarization H.

theorem [11]. This theorem states that (under appropriate ω-stability assumptions) there is a complex gauge transformation making the associated unitary connection self-dual [31], thus providing an identification of $\mathcal{HB}$ with $\mathcal{M}^+ = \{\nabla \in \mathcal{A}[E] : F^+ = 0\}/Aut(E)^u$, where $\mathcal{A}[E]$ is the space of unitary connections on E. Physically, the relation is provided by the quantum equations of motion for g in (4.17):

$$\omega \wedge \left[\nabla_0^{(0,1)}(g^{-1}\nabla_0^{(1,0)}g) - \nabla_0^{(1,0)}\bar{A} + [g^{-1}\nabla_0^{(1,0)}g, \bar{A}]\right] = 0 \quad . \tag{4.19}$$

We view (4.19) as the $(1,1)$ part of the ASD Yang-Mills equation $\hat{F}^+\mathcal{Z} = 0$.

In summary, we get the relation between the generating function for current correlators and quantization of $\mathcal{M}^+$. Following the analogy to WZW_2 we therefore make the following important definition: *The space $H^0(\mathcal{HB}; \mathcal{L}_\omega)$ is the space of (vacuum) 4D conformal blocks for the WZW_4 model.*

Remark. We speculate that the conformal blocks of nontrivial vertex operator correlators associated to divisors can be obtained by quantization of moduli spaces introduced by Kronheimer and Mrowka [32]. These are moduli spaces of ASD connections with singularities along a divisor D such that the limit holonomy around D exists and takes values in a fixed conjugacy class.

4.5. Projected Theory

In this section we let X_4 be an algebraic surface, and denote it as S. Suppose we have a holomorphic map

$$f : S \to C \tag{4.20}$$

where C is an algebraic curve. Correspondingly, there are maps of the moduli spaces, line bundles and their cohomologies:

$$\begin{aligned} &f^* : \mathcal{HB}(C) \to \mathcal{HB}(S) \\ &(f^*)^* : \mathcal{L} \to L^{\otimes k} \\ &(f^*)^* : H^i(\mathcal{HB}(S), \mathcal{L}) \to H^i(\mathcal{HB}(C), L^{\otimes k}) \end{aligned} \tag{4.21}$$

Here L is the standard determinant line bundle over $\mathcal{HB}(C)$, and k equals the integral of ω over a generic fiber of f:

$$k = \int_{f^{-1}(x)} \omega \tag{4.22}$$

[11] which plays the role in four dimensions of the Narasimhan-Seshadri theorem.

This simple observation can be interpreted in more physical terms as follows: Let us define the "projected current" to be:

$$J(x)dx \equiv \int_{f^{-1}(x)} \omega \wedge g^{-1}\partial g = f_*(\omega \wedge g^{-1}\partial g) \tag{4.23}$$

This is a holomorphic 1-form on C and defines an ordinary two-dimensional current algebra with central charge $k = f_*\omega$. Therefore we can proceed to use all the standard constructions from 2D RCFT. For example, we can construct chiral vertex operators which have all the familiar properties of anomalous dimensions, monodromy representations of the braid group, etc.

Remarks

1. In 2D RCFT chiral vertex operators are sometimes heuristically regarded as path-ordered exponentials of a chiral current. In the present context such an identification would read as follows. Let Γ be a curve in C connecting two points x_0 and x_1. Then, given a representation ρ_λ of the group consider the formal relation:

$$\rho_\lambda\left(P\exp\int_\Gamma dx\left\{\int_{f^{-1}(x)}\omega\wedge g^{-1}\partial g\right\}\right) = V_\lambda[D]V_{\lambda^*}[D'] \qquad D = f^{-1}(x_0), D' = f^{-1}(x_1) \tag{4.24}$$

   which might be expected to define 4D chiral vertex operators. In fact, it is a nontrivial problem to define a proper regularization of this expression. [12] From this point of view, anomalous dimensions arise because the projected Green function has *logarithmic* singularities.

2. It is natural to ask what kind of moduli space vertex operators such as those discussed above might correspond to. In a two dimensional theory the insertion of the vertex operator at a point x corresponds to the moduli space of holomorphic bundles $\mathcal{E}$ with parabolic structure at the point x, i.e. there is a flag of subspaces in the fiber $\mathcal{E}|_x$ over the point. The pullback of this bundle to S will produce a holomorphic bundle with parabolic structure along the divisor $f^{-1}(x)$. In this way we again relate correlation functions to the quantization of Kronheimer-Mrowka moduli spaces, since these may be identified with moduli spaces of holomorphic vector bundles with parabolic structure along a divisor [33][34].

[12] Similarly, the proper quantum definition of a Wilson loop in a nonabelian gauge theory is a highly nontrivial problem.

5. Related subjects: Hecke correspondences and Nakajima's algebras

5.1. Correlators and Correspondences

The discussion of the previous section makes contact with some aspects of the Beilinson-Drinfeld "geometric Langlands program" and also with some constructions of Nakajima. Indeed, if $h\bar{\partial}_{\mathcal{E}_1}h^{-1} = \bar{\partial}_{\mathcal{E}_2}$ then the holomorphic vector bundles are isomorphic away from the singularities of h, thus, under good conditions, they will fit into a sequence:

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{S} \to 0 \tag{5.1}$$

for some sheaf $\mathcal{S}$. Conversely, given $\mathcal{S}$ we can form the "geometric Hecke correspondence:"

$$\begin{array}{ccc} & \mathcal{P}_{\mathcal{S}} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathcal{HB}(c_1, c_2) & & \mathcal{HB}(c_1', c_2') \end{array} \tag{5.2}$$

defining, in a standard way, linear operators $(\pi_1)_*(\pi_2^*)$ and $(\pi_2)_*(\pi_1^*)$ on the cohomology spaces. This construction appears in the work of Beilinson and Drinfeld. See also [35][36][37][8].

One example of this construction is familiar in CFT from the study 2D Weyl fermions [24]. The correspondence analogous to (5.2) defined by the sequences

$$\begin{aligned} 0 \to \mathcal{L} \otimes \mathcal{O}(-P) \to & \mathcal{L} \to \mathcal{L} \mid_P \to 0 \\ 0 \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}(Q) \to & [\mathcal{L} \otimes \mathcal{O}(Q)] \mid_Q \to 0 \end{aligned} \tag{5.3}$$

correspond to the effect of inserting $\psi(P), \bar{\psi}(P)$ respectively in the correlation functions which define sections of $H^0(\mathcal{HB}(c_1); \mathrm{DET}\bar{\partial}_{\mathcal{L}})$.

It seems natural to conjecture that if h is singular along divisors or at points then the operators $(\pi_1)_*(\pi_2)^*$ are equivalent to insertion of the vertex operators discussed in sections 4.3, 4.5.

5.2. Nakajima's algebras

In [35] Nakajima has shown that, using the ADHM construction of $U(k)$ gauge theory instantons on ALE spaces $X_n = \widetilde{\mathbb{C}^2/\mathbb{Z}_n}$ one can construct highest weight representations of affine Lie algebras. More precisely, connections on $U(k)$ vector bundles $E \to X_n$ are specified by $c_1(E) \in \Lambda_{root}(SU(n))$, $\mathrm{ch}_2(E) \in \mathbb{Z}_+$, and a flat connection at infinity, i.e., a flat connection on the Lens space $S^3/\mathbb{Z}_n$. The McKay correspondence

gives a 1-1 equivalence between flat $U(k)$ connections on $S^3/\mathbb{Z}_n$ and integrable highest weight representations of $\widehat{SU(n)}_k$ at level k. Let us denote this correspondence as $\rho(\lambda) = \sum \ell_i \rho_i \in Hom(\mathbb{Z}_n \to U(k))/U(k) \leftrightarrow \lambda(\rho) \in HWT\big(\widehat{SU(n)}_k\big)$ where ℓ_i are nonnegative integers assigned to the nodes of the extended Dynkin diagram and $\sum \ell_i = k$.

Nakajima's theorem states that the cohomology of the moduli spaces of "ASD instantons" on X_n is a representation of $\widehat{SU(n)}_k$ at level k. The reason for the quotation marks is explained in the following section. Moreover, $H^p(\mathcal{M}(c_1, \mathrm{ch}_2, \rho(\lambda)))$where $p = \frac{1}{2}\dim \mathcal{M}(c_1, \mathrm{ch}_2, \rho(\lambda))$, is naturally identified with the weight space $\vec{p} = c_1, L_0 = \mathrm{ch}_2(E)$ of the representation $\lambda(\rho)$ of $\widehat{SU(n)}_k$. Nakajima proves this statement using the ADHM construction of instantons. The generators of the affine Lie algebra are defined using a geometric Hecke correspondence, as described above.

5.3. Torsion free sheaves

We now address the question of exactly which moduli space we should use in Nakajima's theorem. [13] The answer is *not* the moduli space of instantons, as is clear by thinking about the case $k = 1$. Indeed, in this case we are discussing $U(1)$ gauge theory and Nakajima's construction gives a level one representation of $\widehat{SU(n)}$. But line bundles have no moduli ! Moreover, for line bundles

$$\mathrm{ch}_2(\mathcal{L}) = \frac{1}{2} c_1(\mathcal{L})^2 \tag{5.4}$$

or, in terms of CFT: $L_0 = \frac{1}{2}\vec{p}^2$. In terms of the Frenkel-Kac construction the line bundles are states with no oscillators.

This problem is resolved when one realizes that the generic solution of the ADHM equations involves a generalization of a line bundle. Indeed, the ADHM equations can be solved rather explicitly for the rank one case, and the moduli space of "line bundles" on $X_n(\vec{\zeta})$ turns out to be:

$$\amalg_{\ell \geq 0} X_n^{[\ell]} \times H^2(X; \mathbb{Z}) \tag{5.5}$$

where $X_n^{[\ell]}$ is the Hilbert scheme of ℓ points on X_n. The proper interpretation of this moduli space is that it is the space of *"torsion free sheaves."* These are sheaves which fit in an exact sequence: $0 \to \mathcal{E} \to \mathcal{L} \to \mathcal{S} \to 0$ where $\mathcal{S}$ is a coherent sheaf supported at

[13] This section represents work done in collaboration with I. Grojnowski. See also [8]. Related results have also been obtained by H. Nakajima [37].

points. [14] In the simplest case where $\mathcal{S}$ is a skyscraper sheaf at $x = y = 0$, $\mathcal{E}$ is given by the $\mathbb{C}[x, y]$ module consisting of functions *vanishing* at $x = y = 0$. In sharp contrast to the situation in $2D$, $\mathcal{E}$ cannot be considered to be a line bundle.

Once we have admitted the necessity for torsion free sheaves we may easily evaluate the "quantum numbers" of $\mathcal{E}$ using the Riemann-Roch- Grothendieck theorem:

$$\begin{aligned} c_1(\mathcal{E}) &= c_1(\mathcal{L}) \\ -\mathrm{ch}_2(\mathcal{E}) &= -\frac{1}{2}(c_1(\mathcal{L}))^2 + \mathrm{ch}_2(\mathcal{S}) = \frac{1}{2}\vec{p}^2 + \ell \end{aligned} \tag{5.6}$$

Here ℓ is, generically, the number of points at which $\mathcal{S}$ has its support. Thus we recover $L_0 = \frac{1}{2}\vec{p}^2 + \ell$.

Another immediate benefit of admitting torsion free sheaves is that we can reproduce the Frenkel-Kac characters from Nakajima's approach. In particular the character is:

$$\mathrm{Tr}_{\mathcal{H}} q^{-\mathrm{ch}_2(\mathcal{E})} = \sum q^{-k(\mathcal{E})} \dim H^{\frac{1}{2}}(X_\zeta^{[\ell]}) = \sum_{p \in \Lambda_{root}} \frac{q^{\frac{1}{2}p^2}}{\eta^r} \tag{5.7}$$

where $H^{\frac{1}{2}}$ is the middle-degree cohomology. The reader will note the similarity to the calculation of [38], which applies the results of [39][40][41] to evaluate the $N = 4$ SYM partition function on $K3$. In general it appears that the compactification by torsion free sheaves, also known as the Gieseker compactification, is the one most relevant to physical theories.

6. Five-Dimensional Viewpoint. Verlinde Formula and Donaldson Invariants

When $X_5 = X_4 \times \mathbb{R}$, X_4 Kähler, we may define the 5D "Kähler-Chern-Simons" theory:

$$\begin{aligned} S = \int_{X^4 \times S^1} Tr(\omega \wedge [AdA + \frac{2}{3}A^3 + dt \wedge \psi \wedge \bar{\psi}] + \\ + dt \wedge [\chi^{2,0} \wedge \bar{\partial}_{\bar{A}} \bar{\psi} + \bar{\chi}^{0,2} \wedge \partial_A \psi + H^{2,0} F^{0,2} + \bar{H}^{0,2} F^{0,2}]) \end{aligned} \tag{6.1}$$

Here A is a 5D gauge field, while $\bar{H}^{0,2} \in \Omega^{0,2}(X_4, \underline{\mathbf{g}})$ etc. are bosonic and $\chi^{2,0} \in \Omega^{2,0}(X_4, \underline{\mathbf{g}})$, $\psi \in \Omega^{1,0}(X_4, \underline{\mathbf{g}})$ are fermionic. The quantization associated with the first line of (6.1) is straightforward and we skip the details. Consequently, on $X_5 = X_4 \times S^1$ we have the partition function $Z_{KCS} = \dim H^0(\mathcal{HB}; \mathcal{L}_\omega)$.

[14] They are called "torsion free" because, as $\mathcal{O}$ modules they have no torsion.

Remarks

1. The Lagrangian (without the fermions) has appeared before in the work of Nair and Schiff [14][15]. It is necessary to introduce the Lagrange multipliers $H, \bar{H}$ for constraints restricting the integral to an integral over $\mathcal{A}^{1,1}$. The constraints $F^{2,0} = F^{0,2} = 0$ are second class, necessitating the fermions. Indeed, $\mathcal{A}^{1,1}$ is invariant under complexified gauge transformations $\mathcal{G}_c$ and $\bar{H}^{0,2}$ fixes this invariance. The gauge fixing introduces the extra terms involving χ, ψ. Put another way, we are simply writing the Poincaré dual, $\eta(\mathcal{A}^{1,1} \hookrightarrow \mathcal{A})$ using a Mathai-Quillen representation. Following the standard procedure one introduces the topological multiplet (A, ψ) and the antighost multiplets: $(\chi^{2,0}, H^{2,0}), (\bar{\chi}^{0,2}, \bar{H}^{0,2})$. [15] The new point here is that there is an extended symmetry in the problem (time-dependent gauge transformations, extended by the $U(1)$, rotating the time loop), and the Poincare dual as well as the whole action is equivariantly closed with respect to this extended symmetry.
2. In the systematic development of 2D RCFT from the 3D CSW point of view one derives the chiral current algebra and representations of WZW_2 from quantization of CSW on a 3-manifold with boundary, e.g. on $B_2 \times \mathbb{R}$ [42][43][44]. In the 5D KCS/WZW_4 theory there are new complications. Nevertheless, similar manipulations allow one to recover the algebra $\kappa(X_3, \underline{\mathbf{g}}, \omega)$ we found in section 2.4 as follows. We let the 4-fold have a boundary, $\partial X_4 = X_3 \neq \emptyset$, e.g., $X_3 = S^3$ or $S^3 \amalg S^3$, and we take the gauge group to be: $\mathcal{G}_1 = \{g(x) \in Aut(P) : g|_{X_3} = 1\}$. This group acts symplectically on $\mathcal{A}$ with moment map
$$\mu(\epsilon) = \int_{X_4} \omega \mathrm{Tr}(\epsilon F) + \int_{X_3} \mathrm{Tr}(\epsilon\omega \wedge A) \tag{6.2}$$
The analog of the moduli space of instantons is the infinite-dimensional symplectic quotient: $\mu^{-1}(0) \cap \mathcal{A}^{1,1}/\mathcal{G}_c \equiv \mathcal{M}^+$. The current algebra is obtained from the algebra of moment maps $\mu(\epsilon)$ and is just $\kappa(X_3, \underline{\mathbf{g}}, \omega)$.

6.1. The Verlinde Formula

The 4D version of the Verlinde formula can be derived by localizing the 5D KCS path integral on the 5-manifold $X_4 \times S^1$, viewing the latter as an example of integration of equivariant differential forms and using a BRST symmetry Q. The calculation is closely related to that of [45][46], as well as the recent calculations of [47]. As in [47] there are two branches of fixed points called A and B. We present here a *preliminary* answer for

[15] For a review of this technology see [3].

the contribution of branch B. Deriving the result for branch A requires further techniques which are under investigation [5].

A fairly extensive calculation, for $\mathcal{E}$ of rank r and $c_1(\mathcal{E}) = 0$, expanding around the fixed points of Q and evaluating the determinants of quadratic fluctuations leads to a formula given by a sum over decomposition into line bundles ("abelianization") $\mathcal{E} \cong \oplus \mathcal{L}_i$. When $b_1(X) = 0$ and $p_g(X) = h^{2,0}(X) = 0$ the formula specializes to:

$$\begin{aligned} \dim H^0(\mathcal{HB}, \mathcal{L}_\omega) = \sum_{\zeta \in \mathrm{Pic}(X^4)^r, [\zeta]\cdot[\zeta] = -k} \int_{\underline{\mathbf{t}} \cap \Delta_+} d\phi e^{-i\frac{1}{2}\langle \phi, [\zeta]\rangle \cdot (\omega + h c_1(X))} \\ (-1)^{\langle \rho, [\zeta]\rangle \cdot c_1(X)} \times \prod_{\alpha > 0} (2\sin(\frac{\phi_\alpha}{2}))^{[\zeta]_\alpha \cdot [\zeta]_\alpha + \frac{1}{6}(c_1(X)^2 + c_2(X))} \end{aligned} \tag{6.3}$$

where $[\zeta] \in H^2(X_4, \mathbb{Z}^r)$ is the collection of first Chern classes of ζ, $a \cdot b = \int_{X_4} a \wedge b$. With the identification of $\mathbb{R}^r$ as the Cartan subalgebra of $SU(r+1)$ we have $\phi_\alpha = \langle \phi, \alpha \rangle$, for the roots α, $\rho = \frac{1}{2}\sum_{\alpha > 0} \alpha$, h is the dual Coxeter number and $\underline{\mathbf{t}} \cap \Delta_+$ is the Weyl alcove. For $SU(2)$ (6.3) may be further simplified to

$$\int_0^{2\pi} d\phi \left(\frac{1}{2\sin \phi/2} \right)^{4k-2} \mathcal{P}(X; k, H, \phi) \tag{6.4}$$

where $\mathcal{P}(X; k, H, \phi)$ is a manifold-dependent distribution

$$\mathcal{P}(X; k, H, \phi) = \sum_{\zeta \in \mathrm{Pic}(X^4), [\zeta]\cdot[\zeta] = -k} (-1)^{[\zeta]\cdot c_1(X)} e^{-i\frac{1}{2}\phi [\zeta] \cdot (H + 2c_1(X))} \tag{6.5}$$

with H the divisor corresponding to the Kähler class.

When $h^{2,0} > 0$ the existence of fermion zeromodes leads to a more compliciated version of (6.3) involving a Grassmann integral over $H^{2,0}(X) \otimes \underline{\mathbf{t}}$. The derivation of (6.3) and its generalization for $h^{2,0} > 0$ will be presented in [5].

Remark. As usual the integral (6.4) is ill-defined because of endpoint divergences. By analogy to previous discussions of nonabelian localization the sense of the contour deformation should be dictated by the sign of the moment map [48] [49]. In the resulting formula, for some choices of manifold X_4, second Chern character k, and embedding H, (6.4) then leads to well-defined integers, even without consideration of the A-branch.

6.2. Semiclassical Limit

A semiclassical limit is of particular interest because it leads to a very computable theory. We set $\omega = k\omega_0$, for some fixed Kähler class ω_0 and take $k \to \infty$. By the index theorem $\dim H^0 = \int \mathrm{ch}(\mathcal{L}) Td(\mathcal{HB})$ [16] so in the large k limit we get:

$$Z_\omega^{KCS} \to k^{\dim \mathcal{M}^+/2} \mathrm{Vol}_{\omega_0}(\mathcal{M}^+) \tag{6.6}$$

On the other hand, taking the limit in the KCS path integral we localize to the space of time-independent fields and get:

$$\begin{aligned} Z_\omega^{KCS} \to k^{\dim \mathcal{M}^+/2} \int \frac{[dAd\psi d\chi dH d\phi]}{\mathrm{vol}\mathcal{G}} \\ \exp\left\{ -\frac{i}{4\pi} \int_{X_4} \mathrm{Tr}\Big[\bar{H}^{0,2} F^{2,0} + \phi\omega_0 \wedge F^{1,1} + H^{2,0} F^{0,2} \right. \\ \left. + \omega_0 \psi \wedge \bar{\psi} + \bar{\chi}^{0,2} \partial_A \psi + \chi^{2,0} \bar{\partial}_A \bar{\psi} \Big] \right\} \end{aligned} \tag{6.7}$$

This theory has been considered before in [49]. The authors of [49] refer to the theory as "holomorphic Yang-Mills theory," (HYM). They derived it in order to produce a QFT expression for the symplectic volume of moduli space. Hyun and Park derived a formula analogous to (6.3) from a localization argument in [47].

Remarks.

1. The volume of the instanton moduli space is one of the Donaldson observables,

$$\langle \exp(\int_{X_4} \omega \wedge \mathrm{Tr}\phi F + \frac{1}{2}\psi\psi) \rangle \tag{6.8}$$

   and for a large class of Kähler surfaces this was computed recently in [50].

2. On hyperkähler manifolds one can integrate out ψ to produce a Lorentz invariant action. Moreover, using the ADHM construction it is possible to write fairly explicit results for the path integral (6.7).

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[16] assuming higher cohomology groups are zero

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