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TOWARDS MIRROR SYMMETRY AS DUALITY FOR TWO DIMENSIONAL ABELIAN GAUGE THEORIES*

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ABSTRACT

Superconformal sigma models with Calabi–Yau target spaces described as complete intersection subvarieties in toric varieties can be obtained as the low-energy limit of certain abelian gauge theories in two dimensions. We formulate mirror symmetry for this class of Calabi–Yau spaces as a duality in the abelian gauge theory, giving the explicit mapping relating the two Lagrangians. The duality relates inequivalent theories which lead to isomorphic theories in the low-energy limit. This formulation suggests that mirror symmetry could be derived using abelian duality. The application of duality in this context is complicated by the presence of nontrivial dynamics and the absence of a global symmetry. We propose a way to overcome these obstacles, leading to a more symmetric Lagrangian. The argument, however, fails to produce a derivation of the conjecture.

Introduction

Two dimensional conformal field theories with $N=2$ supersymmetry have been extensively studied as candidate vacua for perturbative string theory. A particularly interesting class of these consists of supersymmetric sigma models with Calabi–Yau target spaces. These models exhibit a remarkable duality—known as mirror symmetry[1,2,3,4]—which relates two topologically distinct target spaces leading to isomorphic conformal field theories. The duality has the property that classical computations in one model reproduce exact computations—including nonperturbative corrections—in the other. This property has been used to study both the geometry of Calabi–Yau spaces and the properties of the CFT's they determine, with great success. However, a deep understanding of why such a duality exists has been lacking.

The initial observations of mirror pairs in a restricted class of Calabi–Yau spaces

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have been generalized in a set of elegant conjectures by Batyrev and Borisov[5,6,7]. These authors proposed a construction of the mirror partner to a given Calabi–Yau space in a rather broad class (the class of "complete intersections in toric varieties"), and offered some evidence that the space constructed was indeed the mirror. At roughly the same time, Witten[8] noted that for precisely this class of Calabi–Yau spaces, the associated conformal field theory could be obtained as the low-energy approximation to a supersymmetric abelian gauge theory in two dimensions. The coincidence of the two classes suggests the existence of a natural interpretation of the conjectures of Batyrev and Borisov in the context of Witten's model. In this note we present this interpretation, quantifying and making precise some earlier remarks by a number of authors to the effect that mirror symmetry must be a manifestation of electric-magnetic duality in these models. We also present an attempt to establish the equivalence of the two dual models by a modified version of abelian duality, and show how the (present) attempt fails.

1. The Gauged Linear Sigma Model

We begin with a brief review of the gauged linear sigma model (GLSM) construction of Witten[8]. The model is formulated in (2,2) superspace, and requires for its construction the specification of a compact abelian group G, a faithful representation $\rho : G \to U(1)^n$, and a G-invariant polynomial $W(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are coordinates in a complex vector space $\mathbb{C}^n$ on which $U(1)^n$ acts diagonally. To construct the gauged linear sigma model, we begin with n chiral superfields Φ_i (satisfying $\overline{D}_+\Phi_i = \overline{D}_-\Phi_i = 0$) interacting via the holomorphic superpotential $W(\Phi_1, \ldots, \Phi_n)$. The model is invariant under the action of G (via ρ) on $\vec{\Phi}$ and we gauge this action, preserving N=2 supersymmetry, by introducing the $\mathfrak{g}$-valued vector multiplet V with invariant field strength $\Sigma = \frac{1}{\sqrt{2}}\overline{D}_+ D_- V$. This last field is *twisted chiral*, which means that $\overline{D}_+\Sigma = D_-\Sigma = 0$. For each continuous $U(1)$ factor of G we include a Fayet-Illiopoulos D-term and a θ-angle; these terms are naturally written in terms of the complex combination $\tau = ir + \frac{1}{2\pi}\theta$. The resulting Lagrangian density is thus

$$\begin{aligned} \mathcal{L} \;&=\; \int d^4\theta \left(\| e^{R(V)}\vec{\Phi} \|^2 - \frac{1}{4e^2}\|\Sigma\|^2 \right) \\ &\quad + \left(\int d\theta^+ d\theta^- W(\Phi_1, \ldots, \Phi_n) + \text{ c.c. } \right) \\ &\quad + \left(\frac{i}{\sqrt{2}} \int d\theta^+ d\bar{\theta}^- \langle \tau, \Sigma \rangle + \text{ c.c. } \right), \end{aligned} \tag{1}$$

where $R = -i\, d\rho : \mathfrak{g} \to \mathbb{R}^n$ is the derivative of the representation ρ (with a factor of $-i$ to make it real-valued).

Concretely, the general compact abelian group takes the form $G = U(1)^{n-d} \times \Gamma$ where Γ is a product of finite cyclic groups. Choosing a basis for the continuous part we have $n-d$ vector multiplets V_a; the action on the fields is given by $R(V_a)\Phi_i = Q_i^a\Phi_i$ for some integer charges Q_i^a. Notice that the discrete group Γ does not appear explicitly in the Lagrangian, but it does affect the construction of the field theory—the fields are sections of bundles with structure group G. In the case that Γ is nontrivial, we recover in this way an orbifold of another theory (cf. Ref. [8]).

We will be interested in families of such models, parameterized by the coefficients of W and by the instanton factors $q_a := e^{2\pi i\tau_a}$. (There will be a set of complex codimension one in the parameter space along which the model is singular; we will study values of the parameters away from this locus.) A family is thus characterized by the group G and the collection of monomials appearing in W. In order to specify these data, it is convenient to introduce a $u \times n$ matrix P of rank d with nonnegative integer entries, and a factorization $P = ST$ of P as a product of integer matrices S and T, each of rank d. The rows t_α of T can then be used to construct a collection of Laurent monomials $x^{t_\alpha} := \prod x_i^{t_{\alpha i}}$, and the group G is defined to be the largest subgroup of $H = U(1)^n$ which leaves the monomials x^{t_α} invariant. The monomials x^{p_r} defined by the rows of P are then G-invariant by construction, thanks to the relation $p_{ri} = \sum s_{r\alpha}t_{\alpha i}$. Since $p_{ri} \geq 0$ by assumption, we may use these monomials to specify the family of interaction polynomials

$$W(x_1, \ldots, x_n) := \frac{1}{2\pi\sqrt{2}} \sum_{r=1}^{u} c_r x^{p_r} = \frac{1}{2\pi\sqrt{2}} \sum_{r=1}^{u} c_r \prod_{i=1}^{n} x_i^{p_{ri}} . \qquad (2)$$

Alternatively, if we are given G and a family of polynomials W, it is not difficult to reconstruct the matrices P, S, and T. (Actually, S and T are only well-defined up to $(S, T) \mapsto (SL, L^{-1}T)$, with L an invertible integer matrix.)

If we start with only W, as specified by the rows of the matrix P, then the choice of a factorization $P = ST$ amounts to a choice of a subgroup $G \subset U(1)^n$ (which can be less than maximal) under which W is invariant. On the other hand, if we start with only G then the choice of factorized matrix $P = ST$ allows us to specify which of the G-invariant monomials should be included in W. In particular, it is possible to omit some monomials and in this way to study subsets of the maximal set of all G-invariant superpotentials. As we shall see, this possibility is useful. (The assumption made above on the rank of S and T restricts this choice—we must use "enough" monomials to get the rank to be correct—but the restriction is not essential and relaxing it would simply require a more cumbersome notation.)

Under these conditions, the possible choices for factorizations (and hence for G) for a given P are rather limited. In any factorization $P = ST$, the rows of T will generate an integral lattice of rank d, and the rows of P will generate a sublattice, also of rank d. This implies that the quotient

$$\mathrm{rowlattice}(T)/\,\mathrm{rowlattice}(P)$$

is a torsion subgroup of $\mathbb{Z}^n/\,\mathrm{rowlattice}(P)$. There are only finitely many of choices of such a torsion subgroup, once P has been fixed.

2. The R-Symmetry

The gauged linear sigma model is not conformally invariant, and will flow to strong-coupling in the infrared. Generically such a theory would be expected to develop a mass gap, but evidence has been found[8,9] that the theory at the IR fixed point will be nontrivial if we require that the high-energy theory admit a non-anomalous chiral R-symmetry. We consider a right-moving R-symmetry, acting in such a way that θ^+ has charge 1 and θ^- is neutral. Invariance of the kinetic terms requires that the gauge fields V be neutral, and their field strengths Σ hence have charge 1. Invariance of the superpotential interaction requires that the superpotential have charge 1. If we let μ_i denote the R-charge of Φ_i (which may be a rational number) this tells us that

$$\sum_{i=1}^{n} p_{ri}\mu_i = 1 \quad \text{for all } r \ .$$

This chiral symmetry can be anomalous in the presence of the gauge fields. A quick computation[8,10] shows that the anomaly is given by a function on the Lie algebra proportional to $V \mapsto \mathrm{trace}(R(V))$; we require that this vanish identically, i.e., that the symmetry be nonanomalous. Since the action of the continuous part of G on the monomial $x_1 \cdots x_n$ is via $\exp(\mathrm{trace}(R(V)))$, this is the same as requiring that $x_1 \cdots x_n$ be invariant under the continuous part of G (or in our explicit coordinates that $\sum_i Q_i^a = 0$ for all a). This in turn will hold exactly when the vector of exponents $(1, \ldots, 1)$ is a linear combination of the rows of the matrix T, with rational coefficients. (If we had wanted $x_1 \cdots x_n$ to be invariant under all of G, we would have insisted upon integer coefficients.) That is, there is a rational vector $\vec{\lambda}$ such that $\vec{\lambda}^{\mathrm{T}} T = (1, \ldots, 1)$.

Since we are assuming that the $d \times u$ matrix S^{T} has rank d, it is possible to solve the equation $S^{\mathrm{T}}\vec{\nu} = \vec{\lambda}$ for a rational vector $\vec{\nu}$. If we also set $\vec{\mu} = (\mu_1, \ldots, \mu_n)^{\mathrm{T}}$, then we can write the conditions for an unbroken R-symmetry as the existence of $\vec{\mu}$, $\vec{\nu}$ such that

$$\begin{aligned} P\vec{\mu} &= (1, \ldots, 1)^{\mathrm{T}} && (3) \\ \vec{\nu}^{\mathrm{T}} P &= (1, \ldots, 1) \ . && (4) \end{aligned}$$

Finally, using the R-symmetry and calculating as in Ref. [9], one finds that the central charge c of the fixed-point CFT is determined by

$$d - (c/3) = 2\sum_{i=1}^{n} \mu_i = 2\,\vec{\nu}^{\,\mathrm{T}} P \vec{\mu} = 2 \sum_{r=1}^{u} \nu_r \ .$$

It may be useful to have some examples.

Example 1.

Consider 1×1 matrices $(p) = (s)(t)$ with s and t positive integers. Note that Eqs. (3) and (4) are trivially satisfied in this case. Since G is the group which leaves x^t invariant, we must have $G = \mathbb{Z}/(t)$. The interaction polynomial is $W(x) = x^p$, so our model is an orbifold of the Landau-Ginsburg version of a minimal model.

Example 2.

Consider a $u \times 6$ matrix P of rank 5 such that $p_{0i} = p_{r0} = 1$, (letting the i and r indices start at 0 for this example) implementing Eqs. (3) and (4) directly. This leads to a polynomial of the form

$$W(x_0,\ldots,x_5) = x_0\, W'(x_1,\ldots,x_5) = \frac{1}{2\pi\sqrt{2}}\left(c_0\, x_0x_1x_2x_3x_4x_5 + x_0\, f(x_1,\ldots,x_5)\right),$$

where f is a polynomial in 5 variables containing precisely $u{-}1$ monomials. The existence of a kernel for p, an integral generator of which we denote by $(-k,\ell_1,\ldots,\ell_5)^{\mathrm{T}}$, means that W is invariant under a $U(1)$-action on $(x_0,\ldots,x_5)$ with weights $(-k,\ell_1,\ldots,\ell_5)$—this implies that f is quasi-homogeneous of degree k and that $k = \sum \ell_i$. One possible factorization $P = ST$ would then be given by taking the rows of T to be a basis for the $U(1)$-invariant Laurent monomials, and expressing the monomials appearing in f in terms of those. By construction, this leads to $G = U(1)$. In general (if f is a generic quasi-homogeneous polynomial) this is the only possible factorization. For special subfamilies (determined by a subset of the rows of the maximal P) the polynomial is invariant under additional discrete symmetries (the restriction on rank(P) constrains the continuous part), and other factorizations are possible, leading to groups G with a nontrivial discrete part.

As we will see below, the weights we have given specify a weighted projective space $\mathbb{P}^{(\ell_1,\ldots,\ell_5)}$, and $\{f{=}0\} \subset \mathbb{P}^{(\ell_1,\ldots,\ell_5)}$ defines a (singular) Calabi–Yau hypersurface in that space, closely related to the GLSM built from the data $P = ST$. A particularly interesting case is when $u = 6$, and P is 6×6. The description of these models in terms of the matrix P was first given by Candelas, de la Ossa, and Katz[11] in the course of generalizing the Berglund–Hübsch[12] construction.

Example 3.

To make the previous example a bit more concrete, consider a Fermat-type polynomial

$$f(x_1,\ldots,x_5) = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} + x_4^{a_4} + x_5^{a_5}$$

with $\ell_j = k/a_j$ and suppose that $\ell_5 = 1$. Choosing $G = U(1)$ as in example 2 corresponds to the factorization

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & a_2 & 0 & 0 & 0 \\ 1 & 0 & 0 & a_3 & 0 & 0 \\ 1 & 0 & 0 & 0 & a_4 & 0 \\ 1 & 0 & 0 & 0 & 0 & a_5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & a_1 & 0 & 0 & 0 \\ 1 & 0 & a_2 & 0 & 0 \\ 1 & 0 & 0 & a_3 & 0 \\ 1 & 0 & 0 & 0 & a_4 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & k \\ 0 & 1 & 0 & 0 & 0 & -\ell_1 \\ 0 & 0 & 1 & 0 & 0 & -\ell_2 \\ 0 & 0 & 0 & 1 & 0 & -\ell_3 \\ 0 & 0 & 0 & 0 & 1 & -\ell_4 \end{pmatrix} .$$

On the other hand, to maximize the group G we should use the entire row space of P as the row space of T which leads to the factorization

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & a_2 & 0 & 0 & 0 \\ 1 & 0 & 0 & a_3 & 0 & 0 \\ 1 & 0 & 0 & 0 & a_4 & 0 \\ 1 & 0 & 0 & 0 & 0 & a_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ k & -\ell_1 & -\ell_2 & -\ell_3 & -\ell_4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & a_2 & 0 & 0 & 0 \\ 1 & 0 & 0 & a_3 & 0 & 0 \\ 1 & 0 & 0 & 0 & a_4 & 0 \end{pmatrix} .$$

This corresponds to an orbifold of the previous model; in fact, this is the quotient found by Greene and Plesser[4] to correspond to the mirror manifold. Thus we can reproduce the construction of Ref. [4] in this context by appropriate choices of factorization.

3. The Low-Energy Limit

As explained in detail in Ref. [8], the low-energy limit of this theory can be described explicitly and—when the central charge is an integer—often coincides with a sigma-model on a Calabi–Yau space (for appropriate values of the parameters).

To study the low-energy limit one begins by mapping out the space of classical vacua of the theory. To this end, we first solve the algebraic equations of motion for the auxiliary fields D_a (in the vector multiplets) and F_i (in the chiral multiplets)

$$D_a = -e^2 \left(\sum_{i=1}^{n} Q_i^a |\phi_i|^2 - r_a \right) \tag{5}$$

$$F_i = -\frac{\partial W}{\partial \phi_i} . \tag{6}$$

The potential energy for the bosonic zero modes is then

$$U = \frac{1}{2e^2} \sum_{a=1}^{n-d} D_a^2 + \sum_{i=1}^{n} |F_i|^2 + \sum_{a,b} \bar\sigma_a \sigma_b \sum_{i=1}^{n} Q_i^a Q_i^b |\phi_i|^2 ,$$

where ϕ_i, σ_a are the lowest components of Φ_i, Σ_a respectively. The space of classical vacua is the quotient by G of the set of zeros of U.

Neglecting for a moment the superpotential, the space of solutions (setting $\sigma = 0$) is $D^{-1}(0)/G$. For values of the instanton factors q_a in a suitable range this is a toric variety of dimension d. In the "geometric" case in which $\vec{\mu}$ has N 1's and $n-N$ 0's, this variety can be recognized as the total space of a sum of N line bundles over a *compact* toric variety of dimension $d - N$. The equations $F_i = 0$ then determine (for generic values of the parameters c_r) a complete intersection subvariety of codimension N in the base space, homologous to the intersection of the divisors associated to the N line bundles. The condition in Eq. (4) implies that this subvariety is Calabi–Yau.

The fixed-point CFT is given by the nonlinear sigma model with this target space. The moduli of this CFT are the marginal operators in Eq. (1). Naïvely, both the q_a and the c_r would appear to be marginal couplings. The latter, however, are not all independent. As is well known, some of them can be absorbed in rescalings of the fields Φ_i. The true moduli are thus the q_a and the scaling-invariant combinations of the c_r. There will in general be other marginal operators in the model which do not appear explicitly in the Lagrangian; we restrict our attention to the subspace of those that do. Of course, the linear (or more accurately, toric) structure with which we have endowed our moduli space is an artifact of our description. In particular, it is consistent with the natural complex structure that this moduli space carries intrinsically, but bears no relation to the Kähler structure determined by the Zamolodchikov metric. In other words, the q_a and c_r are not the "special" coordinates on this space.

For example, in example 2 above in which $G = U(1)$, the D-term equation is

$$-k\,|x_0|^2 + \sum_{i>0} \ell_i |x_i|^2 - r = 0,$$

so when $r > 0$ we cannot have $x_i = 0$ for all $i > 0$. The space of solutions $D^{-1}(0)/G$ can be recognized as the total space of the canonical bundle over $\mathbb{P}^{(\ell_1,\ldots,\ell_5)}$. If we impose the F-term equation as well

$$\sum_{i=0}^{5} \left| \frac{\partial W}{\partial x_i} \right|^2 = |W'|^2 + |x_0|^2 \sum_{i=1}^{5} \left| \frac{\partial W'}{\partial x_i} \right|^2 = 0,$$

then for a generic choice of the coefficients of W (away from a codimension-one subspace we avoid as promised above) the space of solutions is given by $x_0 = 0$ and $W'(x) = 0$, yielding a Calabi–Yau hypersurface in $\mathbb{P}^{(\ell_1,\ldots,\ell_5)}$. Further study shows that the low-energy excitations are tangent to this, so that the low-energy theory is a nonlinear sigma model with this Calabi–Yau target space.

4. Mirror Symmetry

We are now in a position to state the mirror symmetry conjectures for this class of models. Mirror symmetry relates two Calabi–Yau manifolds which lead to isomorphic

CFT's when used as target spaces for supersymmetric nonlinear sigma models. The mirror isomorphism reverses the sign of the right-moving R-symmetry; many of the fascinating properties of mirror pairs can be traced to this feature. Given a GLSM family determined by $P = ST$ we will construct the mirror family in the same class of models. We incorporate the sign change on R by writing the dual model using *twisted* chiral matter fields coupled to twisted gauge multiplets (with *chiral* field strengths). We will call such a model a *twisted* gauged linear sigma model. As above, we characterize the family of models by a factorized matrix of nonnegative integers $\widehat{P} = \widehat{S}\widehat{T}$. We use this data to construct a twisted superpotential $\widehat{W}$ and write a Lagrangian density (compare Eq. (1))

$$\begin{aligned}
\mathcal{L} &= \int d^4\theta \left(-\|e^{\widehat{R}(\widehat{V})}\vec{\widehat{\Phi}}\|^2 + \frac{1}{4e^2}\|\widehat{\Sigma}\|^2 \right) \\
&\quad + \left(\int d\theta^+ d\bar{\theta}^- \widehat{W}(\widehat{\Phi}) + \text{ c.c. } \right) \\
&\quad + \left(\frac{i}{\sqrt{2}} \int d\theta^+ d\theta^- \langle \hat{\tau}, \widehat{\Sigma} \rangle + \text{ c.c. } \right),
\end{aligned} \tag{7}$$

where $\widehat{\Sigma} = -\frac{1}{2\sqrt{2}}\bar{D}_+\bar{D}_-\widehat{V}$ is the (chiral) field strength.

The *mirror conjecture* states that if we set

$$\begin{aligned}
\widehat{P} &= P^{\mathrm{T}} \\
\widehat{S} &= T^{\mathrm{T}} \\
\widehat{T} &= S^{\mathrm{T}}
\end{aligned} \tag{8}$$

the two families of CFT's defined by the infrared limits of the two linear models are isomorphic. This is essentially just a translation of the conjecture of Batyrev and Borisov[7], versions of which have appeared in Refs. [13,11]. In fact, the statement given here generalizes the conjecture somewhat since the factorization makes possible a map between subfamilies.

One can make a more precise statement of the conjecture—specifying explicitly the map between the two parameter spaces which relates isomorphic low-energy limits

$$\begin{aligned}
q_a &= \prod_{r=1}^{u} \hat{c}_i^{Q_i^a} \\
\widehat{q_{\hat{a}}} &= \prod_{i=1}^{n} c_r^{\widehat{Q}_r^{\hat{a}}} .
\end{aligned} \tag{9}$$

Note that as expected only the true moduli appear on the right-hand side of Eq. (9). The asymptotic limits of Eq. (9) constitute the *monomial-divisor mirror map*, first proposed in Refs. [14,15] (see also Ref. [16]) and extended in Refs. [13,17]. The ability to write a global version of this map and not just an asymptotic form is related to the coordinates we use on the moduli space (for a full discussion of this including a proof of Eq. (9) for a class of examples see Ref. [10]).

We note that this statement of the conjecture is very reminiscent of recent results on duality in four-dimensional supersymmetric gauge theories[18,19,20]. Two distinct gauge theories with nontrivial dynamics lead to the same low-energy physics; further, instanton effects in one model are reproduced by classical physics in the other.

In example 3 above, it should be clear that this construction interchanges the two choices of factorization, leaving the (symmetric) matrix P unchanged. As mentioned above, this reproduces the orbifold construction of the mirror in Ref. [4], clearly and explicitly demonstrating how the general construction reduces to this upon restricting to a subfamily of all possible W—precisely the subfamily invariant under the maximal discrete group.

5. Abelian Duality

The formulation we have given for the mirror conjecture in the context of the GLSM immediately suggests a relation to abelian duality. We recall that given a theory with an abelian continuous global symmetry, one can use duality to obtain an equivalent theory. The dual model is obtained by gauging the global symmetry, introducing Lagrange multipliers which constrain the connection to be pure gauge (ensuring that the theory is actually unchanged). One must then eliminate the original degrees of freedom, in the process generating a nontrivial effective action for the Lagrange multipliers, which become the fundamental degrees of freedom for the dual model. Classically this is accomplished by gauge-fixing. In a theory with $N{=}2$ supersymmetry and chiral charged fields, the Lagrange multipliers mentioned above will be *twisted* chiral fields, and will appear in the twisted superpotential as $\widehat{\Lambda}\Sigma$. Thus the dual variables will naturally be twisted chiral[21], as expected for the mirror model. This idea was pursued in Refs. [22,23,24,25], and in special cases leads to an interpretation of mirror symmetry as abelian duality. In the general case, however, the approach runs into difficulties. The most obvious one is that a generic Calabi–Yau manifold M has no isometries, hence the associated two dimensional CFT lacks suitable global symmetries.

If this approach is to lead to the equivalence of GLSM models which is tantamount to the mirror conjecture, some modification will be required. The two linear models are *not* equivalent; the conjecture implies only that they become equivalent in the extreme low-energy limit. This is consistent with the fact that the model has nontrivial dynamics. As discussed above this is similar to the recent discoveries in four-dimensional supersymmetric gauge theories.

In the context of the GLSM, there is a natural symmetry group one would attempt to use. This is the group $H = U(1)^n$ acting by phases on the fields Φ_i. The difficulty, of course, is that this symmetry is explicitly broken by the superpotential Eq. (2) to the subgroup G. Since this symmetry is gauged in Eq. (1), the resulting model has no global symmetries at all (except the R-symmetry which we cannot gauge).[a] We can attempt to overcome this obstacle by recasting the symmetry-breaking terms as anomalies. To implement this idea, introduce a set of twisted chiral fields $\widehat{\Psi}_r$ into the model, coupled to twisted gauge multiplets $\widehat{V}_r$ gauging the group $\widehat{H} = U(1)^u$ which acts by phases, i.e. consider the additional term in the kinetic energy

$$\mathcal{L}_{\widehat{k}} = \int d^4\theta \left(-\|\bar{\widehat{\Psi}} e^{\widehat{V}}\|^2 + \frac{1}{4e^2}\|\widehat{\Sigma}\|^2 \right) . \tag{10}$$

We then replace the superpotential Eq. (2) by

$$W_s = \frac{1}{2\pi\sqrt{2}} \sum_{r=1}^{u} \widehat{\Sigma}_r \left[\log\left(c_r \prod_{i=1}^{n} \Phi_i^{p_{ri}} \right) + 1 \right] . \tag{11}$$

In this form it is clear that H is broken by $\widehat{H}$ anomalies, as desired. Note that this interaction does not break the R-symmetry. However, the couplings c_r have nonzero beta functions, which will cause them to grow large, so the $\widehat{H}$ gauge theory is strongly coupled at low energies. In this limit, as discussed in Refs. [8,10], this sector of the model is in a confining phase, in which the lowest component $\widehat{\sigma}$ of $\widehat{\Sigma}$ gets a nonzero expectation value. The charged fields are then all massive, with masses of order this expectation value, and the light degrees of freedom are in the $\widehat{\Sigma}$ multiplet. The effective action for these can be reliably computed at one-loop order. Integrating out the $\widehat{\Psi}$'s leads to the effective superpotential

$$W_{\text{eff}} = \frac{1}{2\pi\sqrt{2}} \sum_{r=1}^{u} \widehat{\Sigma}_r \left[\log\left(c_r \prod_{i=1}^{n} \Phi_i^{p_{ir}} \right) + 1 - \log \widehat{\Sigma}_r \right] . \tag{12}$$

In the effective theory we can treat $\widehat{\Sigma}$ as a chiral field; since there are no charged fields we can forget its relation to a gauge symmetry. At low energy, we can eliminate $\widehat{\Sigma}$ from Eq. (12) to get $\widehat{\Sigma}_r = c_r \prod_{i=1}^{n} \Phi_i^{p_{ir}}$. When substituted in Eq. (12), this leads to Eq. (2) as the effective superpotential for Φ. Thus at low energies this model is equivalent to the original GLSM, while the symmetry-breaking terms are explicitly exhibited as anomalies.

We now wish to perform a duality transformation with respect to the anomalous symmetry. Gauging an anomalous symmetry appears inconsistent, but in performing a duality transformation an anomalous abelian symmetry can be restored by assigning transformation properties (under the twisted symmetries) to the Lagrange multipliers.[26] In the case at hand, gauging the global symmetry means that the gauge group becomes all of H. We should also introduce Lagrange multipliers, transforming under $\widehat{H}$, to constrain the field strength to lie in $\mathfrak{g} \subset \mathfrak{h}$. Classically, integrating

[a]This once led E. Witten to describe the problem of understanding mirror symmetry in this model as the question "How to perform duality on a non-symmetry?"

these out will reproduce the original model. This is not true quantum mechanically, however, as is most easily seen by introducing a small kinetic term for the Lagrange multipliers, and considering the limit as this term is removed. It then becomes clear that the contribution of these to the one-loop effective superpotential for $\widehat{\Sigma}$ is constant in the limit. Introducing these extra fields would thus destroy the one-loop calculation that led to Eq. (12).

This suggests that we consider a related Lagrangian, in which the twisted chiral matter is an accurate "reflection" of the chiral matter. This defines a model that is manifestly mirror-symmetric. Our proposal for this new model contains chiral fields $\vec{\Phi}$ and twisted chiral fields $\hat{\vec{\Phi}}$, and (twisted) gauge multiplets gauging the entire group H ($\widehat{H}$). We begin with the Lagrangian density

$$\begin{aligned}\mathcal{L} &= \int d^4\theta \left(\sum_i \left(J_i - \frac{1}{4e^2}|\Sigma_i|^2 \right) - \sum_r \left(\widehat{J}_r - \frac{1}{4e^2}|\widehat{\Sigma}_r|^2 \right) + \frac{1}{2\pi} \sum_{ri} p_{ri} \log J_i \log \widehat{J}_r \right) \\ &\quad + \left(\frac{i}{\sqrt{2}} \int d\theta^+ d\theta^- \langle \widehat{\Sigma}, \widehat{\tau} \rangle + \mathrm{c.c} \right) + \left(\frac{i}{\sqrt{2}} \int d\theta^+ d\bar{\theta}^- \langle \Sigma, \tau \rangle + \mathrm{c.c} \right) , \qquad (13)\end{aligned}$$

where $J_i = |\Phi_i e^{V_i}|^2$ and likewise for $\widehat{J}_r$.

This action is manifestly invariant under $H \times \widehat{H}$. It is classically equivalent to the action we obtain by integration by parts,

$$\begin{aligned}\mathcal{L}_{\mathrm{eq}} &= \int d^4\theta \left(\sum_i \left(J_i - \frac{1}{4e^2}|\Sigma_i|^2 \right) - \sum_r \left(\widehat{J}_r - \frac{1}{4e^2}|\widehat{\Sigma}_r|^2 \right) + \frac{2}{\pi} \sum_{ri} p_{ri} V_i \widehat{V}_r \right) \\ &\quad + \left(\int d\theta^+ d\theta^- W(\widehat{\Sigma}, \Phi) + \mathrm{c.c} \right) + \left(\int d\theta^+ d\bar{\theta}^- \widetilde{W}(\Sigma, \hat{\Phi}) + \mathrm{c.c} \right) , \qquad (14)\end{aligned}$$

where the modified superpotentials are

$$\begin{aligned} W(\widehat{\Sigma}, \Phi) &= \frac{1}{2\pi\sqrt{2}} \sum_{r=1}^{u} \widehat{\Sigma}_r \left[\log \left(\widehat{q}_r \prod_{i=1}^{n} \Phi_i^{p_{ri}} \right) + 1 \right] \\ \widetilde{W}(\Sigma, \hat{\Phi}) &= \frac{1}{2\pi\sqrt{2}} \sum_{i=1}^{n} \Sigma_i \left[\log \left(q_i \prod_{r=1}^{u} \hat{\Phi}_r^{-p_{ri}} \right) + 1 \right] , \qquad (15)\end{aligned}$$

We see that the conditions for a nonanomalous R-symmetry are precisely Eqs. (3) and (4). When these hold we expect to find at low energies an $N{=}2$ superconformal field theory where the R-symmetry defined above becomes the chiral $U(1)$ contained in the superconformal algebra. We will use the second formulation as our definition of the quantum theory. (In the presence of instantons the integration by parts requires the consideration of boundary terms, which are nontrivial.)

It is worthwhile noting the way in which Eq. (14) manages to be gauge-invariant despite the manifestly non-invariant interactions. Under a gauge transformation (in $H \times \widehat{H}$) the variation of Eq. (15) is cancelled, up to a total derivative, by the variation of the $V\widehat{V}$ term. Thus the full action is invariant under gauge transformations approaching the identity at infinity, while transformations with constant parameter are still anomalous. We note that this cancellation holds precisely when the exponents in the two superpotentials are related as in Eq. (15). In a derivation of mirror symmetry along these lines this would be the origin of the first part of Eq. (8).

At this point one still needs to show that the theory defined by Eq. (14) is equivalent to Eq. (1). This can be achieved using the fact that the coefficients q_i, $\widehat{q}_r$ once more contain redundant deformations which can be undone by field rescalings (in fact this is precisely the holomorphic extension of the statement above about anomalies). One can perform a redundant deformation of the model to find a region in parameter space in which the $\widehat{\Sigma}, \widehat{\Phi}$ system is in a confining phase and the one-loop approximation that led to Eq. (12) is valid. One can also show that in this region the integration over $\widehat{\Phi}$ will lead to constraints on Σ, restricting the gauge group to G. In fact, the last two equations of Eq. (8) arise naturally in this context as well.

The manifest mirror symmetry would then lead one to expect that Eq. (14) will also be equivalent to Eq. (7) under the conditions Eqs. (8) and (9). To see this one performs redundant deformations to a region in parameter space in which the Σ, Φ system confines and follows a "mirror" version of the argument sketched above. This naturally incorporates all of the conditions in Eqs. (8) and (9). We hasten to point out however that Eq. (14) is not in fact mirror symmetric. The reason for this is the sign in the second of Eq. (15).[b] Unexpectedly, this sign, which is required for gauge invariance since it follows by integration by parts from Eq. (13), cannot be removed (it is related to the relative sign of the kinetic terms for the chiral and twisted chiral fields). Similar signs appear often in discussions of duality, and are usually harmless. In the case at hand, however, the sign multiplies a logarithm, and its effect is to lead, at least superficially, to a twisted superpotential polynomial in $\widehat{\Phi}^{-1}$ rather than $\widehat{\Phi}$ after performing the "mirror" argument.

This would seem to shatter the hopes for an understanding of mirror symmetry for these models along the lines presented here. However, the model we present does seem to incorporate naturally many of the features required for a derivation of the conjectures in section four. It is possible that some modification of this model will lead to the desired result. Future study will tell if this is indeed the case.

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[b]This sign was inadvertently omitted in earlier stages of this work, leading to conclusions—reported in several talks by the authors—differing from those presented here.

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