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S-Duality in N=4 Yang-Mills Theories [1]

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ABSTRACT

Evidence in favor of $SL(2,Z)$ S-duality in $N=4$ supersymmetric Yang-Mills theories in four dimensions and with general compact, simple gauge groups is presented.

[1] Talk presented by A. Giveon at the *Strings '95* conference, March 13-18, 1995, USC, and the *Trieste Conference on S-Duality and Mirror Symmetry*, June 5-9, 1995.

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1 Introduction

This talk is based on refs. [1, 2].

Electric-magnetic duality appears already in classical Maxwell's equations with magnetic monopoles. Here we will present evidence in favor of $SL(2,Z)$ S-duality – which includes, in particular, the electric-magnetic duality – in $N=4$ supersymmetric Yang-Mills (YM) theories with general, simple gauge groups.

Electric-magnetic duality was conjectured by Montonen-Olive (MO) [3] for gauge theories (although the mass spectrum is in general difficult to compute, due to quantum corrections). A remarkable simplification happens in $N=4$ supersymmetric gauge theories [4]: the supersymmetry algebra implies exact results for masses and charges of "short multiplets" – supersymmetry multiplets containing spin ≤ 1. The masses are given in terms of the electric coupling constant, $g_e \equiv g$, and the magnetic one, $g_m = 4\pi/g$, by

$$M^2 \sim p^2 g_e^2 + q^2 g_m^2, \qquad p, q \in Z. \tag{1.1}$$

Here M^2 is invariant under $g_m \leftrightarrow g_e$ together with $p \leftrightarrow q$. Therefore, $N=4$ is the most likely theory to verify the MO conjecture [4, 5]. For general gauge groups, G, the electric-magnetic duality transformation is expected to take the form [6]:

$$g \to \frac{4\pi}{g}, \qquad G \to \hat{G}, \tag{1.2}$$

where $\hat{G}$ is the dual gauge group.

$SL(2,Z)$-duality was recognized in lattice models (with non-zero theta parameter, $\theta \neq 0$) [7], and conjectured in string theory [8, 9, 10]. A version of S-duality is used to compute exact results in $N=2$ gauge theories [11], and a version of electric-magnetic duality appears also in $N=1$ gauge theories [12].

Important new evidence for S-duality was found in [13, 14, 15]. Sen [13] found stable $(p,q) = (2n+1, 2)$ states in addition to the well-known (1,0) electrically charged states, (0,1) monopoles, and the (1,1) dyons. Recently, it has been shown in ref. [14] that *all* states with p, q relatively prime do indeed exist. A strong-coupling test of S-duality was presented by Vafa and Witten [15]. They showed that a topological twisted version of $N=4$ gauge theories has an S-dual partition function on various manifolds.

Since this is a string theory conference, we should mention that if S-duality is a fundamental symmetry of string theory, it will explain its appearance in gauge theories. The outline of the talk is the following. In section 2, we briefly review the $N=4$ supersymmetric YM theory. In section 3, we will present the 't Hooft box with twisted boundary conditions [16], generalized to arbitrary compact G, the free energies, and the S-duality conjecture. In section 4, we will present the result of computing the leading infrared (IR) divergent term of the free energy, and will discuss its properties. In section 5, we will present the S-duality transformations of the free energies. Finally, we will conclude with a few remarks in section 6.

2 $N = 4$ Supersymmetric YM Theories

An $N = 4$ supersymmetric YM theory is the flat limit ($\alpha' \to 0$) of heterotic strings compactified to $D = 4$ on a torus[6]. S-duality holds order by order in α' and, therefore, if it is a symmetry of string theory it is also a symmetry at the $\alpha' \to 0$ limit.

$N = 4$ supersymmetric YM theory in $D = 4$ is completely determined by the gauge group. The fields in the Lagrangian, L, form a supermultiplet:

$$\begin{gathered}(A^a_\mu,\ \ \lambda^a_I,\ \ \phi^a_{IJ})\\ \mu = 1,2,3,4, \qquad I,J = 1,2,3,4, \qquad a = 1,...,\dim G.\end{gathered} \tag{2.1}$$

All the fields in (2.1) are in the adjoint representation of the gauge group. The supermultiplet contains a gauge field (spin-1), A^a_μ (μ is a space-time vector index and a is a group index of the adjoint representation), four Weyl spinors (spin-1/2), λ^a_I (I is the so-called "extension index," in the 4 of $SU(4)$, representing the four supersymmetry charges), and six scalars (spin-0), ϕ^a_{IJ}, which obey the condition: $2\phi^a_{IJ} = \epsilon_{IJKL}(\phi^a_{KL})^*$. The Lagrangian takes the form

$$\begin{aligned} L \ = \ & \frac{1}{4\pi}\mathrm{Re}\, S\Big[\frac{1}{2}F^a_{\mu\nu}F^{a\,\mu\nu} + \bar{\lambda}^{a\,I}\not{D}\lambda^a_I + D_\mu\phi^{a\,IJ}D^\mu\phi^a_{IJ} \\ & + f_{abc}\bar{\lambda}^{a\,I}\phi^b_{IJ}\lambda^{c\,J} + f_{abc}f_{ade}\phi^b_{IJ}\phi^{c\,JK}\phi^d_{KL}\phi^{e\,LI}\Big] \\ & - \frac{i}{8\pi}\mathrm{Im}\, S F^a_{\mu\nu}\tilde{F}^{a\,\mu\nu}. \end{aligned} \tag{2.2}$$

Here $\phi^{a\,IJ} \equiv (\phi^a_{IJ})^*$, $2\tilde{F}^{a\,\mu\nu} = \epsilon^{\mu\nu\sigma\rho}F^a_{\sigma\rho}$, and f_{abc} are the structure constants of G. From L one reads:

$$S = \alpha^{-1} + ia, \qquad \alpha = \frac{g^2}{4\pi}, \qquad a = \frac{\theta}{2\pi}, \tag{2.3}$$

where g is the coupling constant and θ is a theta parameter. The theory is scale invariant: $\beta(g) = 0$, and the scalar potential has flat directions when $\langle\Phi\rangle \in$ Cartan Sub-Algebra (CSA), and is non-renormalized, even non-perturbatively [18].

Our aim is to find appropriate gauge invariant quantities which are simple enough to be calculable, yet non-trivial, i.e., they carry some dynamical information about the theory, and to test S-duality. One possibility is to follow 't Hooft strategy where the non-Abelian equivalent of the electric and magnetic fluxes are defined.

3 't Hooft Box with Twisted Boundary Conditions, the Free Energy, and the S-Duality Conjecture

't Hooft strategy for $SU(N)$ [16] can be applied to any gauge theory which contains elementary fields in the adjoint representation and, in particular, to $N = 4$ YM theories. The idea (for

[6]Recently, in ref. [17], it was claimed that there exist $D = 4$, $N = 4$ heterotic backgrounds that are *not* toroidal compactifications. Such backgrounds would admit, in particular, non-simply-laced gauge groups.

$SU(N)$) is to write Euclidean functional integrals in a box of sides (a_1, a_2, a_3, a_4) with twisted boundary conditions: $n_{\mu\nu} \in Z_N$ (the center of $SU(N)$)), $n_{\nu\mu} = -n_{\mu\nu}$. To explain the boundary conditions and generalize to any compact (simple) G we need some algebra and notations.

The notations are:

- $G \equiv$ a compact, simple Lie group.
- $\tilde{G} \equiv$ the universal covering group of G.
- $G = \tilde{G}/K$, $K \subseteq C$, $C \equiv Center(\tilde{G})$.
- $\mathcal{G} \equiv$ the Lie algebra of G.
- $\hat{G} \equiv$ the dual group of G.
- $\hat{\mathcal{G}} \equiv$ the dual Lie algebra, i.e., the Lie algebra of $\hat{G}$.
- $\Lambda_R \equiv \Lambda_R(\mathcal{G})$, the root lattice of $\mathcal{G}$, with normalization (long root)$^2 = 2$.
- $\Lambda_W \equiv \Lambda_W(\mathcal{G})$, the weight lattice of $\mathcal{G}$.
- $\hat{\Lambda}_R = (\Lambda_W(\mathcal{G}))^{dual}$.
- $\hat{\Lambda}_W = (\Lambda_R(\mathcal{G}))^{dual}$.
- $\hat{\Lambda}_{L,R}(\mathcal{G}) = N(\mathcal{G})\Lambda_{L,R}(\hat{\mathcal{G}})$, where $N(\mathcal{G}) = 1$ if G is simply-laced, and $N(\mathcal{G}) = \sqrt{2}$ if G is non-simply-laced.
- The group G has a weight lattice of representations which is a sub-lattice of Λ_W: $G = \tilde{G}/K$ $\Rightarrow \Lambda_W(G) = \Lambda_W/K$.
- The dual group $\hat{G}$ has a weight lattice dual to the weight lattice of G: $\Lambda_W(\hat{G}) = \Lambda_W(G)^{dual}$.
- $\hat{\Lambda}(G)_{R,W} = N(\mathcal{G})\Lambda(\hat{G})_{R,W}$, where $N(\mathcal{G}) = 1$ if G is simply-laced, and $N(\mathcal{G}) = \sqrt{2}$ if G is non-simply-laced.
- For G simply-laced: $\hat{\mathcal{G}} = \mathcal{G}$.
- For G non-simply-laced: $\mathcal{G} = so(2n+1) \Leftrightarrow \hat{\mathcal{G}} = sp(2n)$. (The Lie algebrae of G_2 and F_4 are self-dual.)

The center, C, of $\tilde{G}$ is:

$$C = \{e^{2\pi i \hat{w}\cdot T} | \hat{w} \in \hat{\Lambda}_W/\hat{\Lambda}_R\}. \tag{3.1}$$

Here $\hat{w}$ is a vector with components $\hat{w}^P$, $P = 1, ..., r = \text{rank}\, G$, and $\{T_P\}_{P=1,...,r}$ are the generators in the CSA. A weight $w = (w_1, ..., w_r)$ is the eigenvalue of $(T_1, ..., T_r)$ corresponding to one common eigenvector in a single valued representation of G:

$$T_P V_w = w_P V_w, \qquad w \in \Lambda_W. \tag{3.2}$$

We now want to evaluate the Euclidean functional integral in a box of sides a_μ, with twisted boundary conditions in the center $\hat{k}_i, \hat{m}_i \in \hat{\Lambda}_W/\hat{\Lambda}_R \simeq C$, $i, j = 1, 2, 3$ (space indices):

$$W[\hat{\mathbf{k}}, \hat{\mathbf{m}}] = \int [dA^a_\mu \, d\lambda^a_I \, d\phi^a_{IJ}] \exp(-\int d^4x L). \tag{3.3}$$

The center elements $\hat{\mathbf{k}}, \hat{\mathbf{m}}$ are defined through the boundary conditions as follows. The boundary conditions for all bosonic (fermionic) fields are periodic (anti-periodic) up to a gauge transformation:

$$\Phi(x + a_\mu e_\mu) = (-)^F \Omega_\mu(x) \Phi(x), \tag{3.4}$$

where e_μ is a unit vector in the μ direction, and repeated indices are not summed. Φ and $\Omega\Phi$ denote generically a field of the supermultiplet (2.1) and its gauge transform under Ω, respectively; F is the fermion number. Going from x to $x + a_\nu e_\nu + a_\mu e_\mu$, $\mu \neq \nu$, in two different ways – either in the ν direction first and then in the μ direction or vice-versa – implies the consistency conditions:

$$\begin{aligned} \Omega_\mu(x + a_\nu e_\nu)\Omega_\nu(x) &= \Omega_\nu(x + a_\mu e_\mu)\Omega_\mu(x) z_{\mu\nu}, \\ z_{\mu\nu} &\in C, \end{aligned} \tag{3.5}$$

$$z_{\mu\nu} \equiv z_{\hat{w}_{\mu\nu}} = e^{2\pi i \hat{w}_{\mu\nu} \cdot T}, \qquad \hat{w}_{\mu\nu} \in \frac{\hat{\Lambda}_W}{\hat{\Lambda}_R}, \qquad \hat{w}_{\nu\mu} = -\hat{w}_{\mu\nu}. \tag{3.6}$$

The elements $\hat{m}_i$ in $W[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$ are defined by the twists in the spatial directions:

$$\hat{m}_i \equiv \frac{1}{2}\epsilon_{ijk}\hat{w}_{jk}, \qquad i, j, k = 1, 2, 3. \tag{3.7}$$

$\hat{m}_i$ are interpreted as non-Abelian "magnetic fluxes" [16]. The elements $\hat{k}_i$ in $W[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$ are defined by the twists in the time and space directions:

$$\hat{k}_i \equiv \hat{w}_{4i}, \qquad i = 1, 2, 3. \tag{3.8}$$

$\hat{k}_i$ are interpreted as the dual "electric fluxes." The non-Abelian electric fluxes, $e_i \in \Lambda_W/\Lambda_R$, are linked to $\hat{k}_i$ by the equation:

$$e^{-\beta F[\mathbf{e}, \hat{\mathbf{m}}]} = \frac{1}{N^3} \sum_{\hat{\mathbf{k}} \in (\hat{\Lambda}_W/\hat{\Lambda}_R)^3} e^{2\pi i \mathbf{e} \cdot \hat{\mathbf{k}}} W[\hat{\mathbf{k}}, \hat{\mathbf{m}}]. \tag{3.9}$$

Here $\beta \equiv a_4$ is the inverse temperature, $F[\mathbf{e}, \hat{\mathbf{m}}]$ is the free energy of a configuration with electric flux $\mathbf{e}$ and magnetic flux $\hat{\mathbf{m}}$:

$$\mathbf{e} = (e_1, e_2, e_3), \;\; e_i \in \Lambda_W/\Lambda_R, \qquad \hat{\mathbf{m}} = (\hat{m}_1, \hat{m}_2, \hat{m}_3), \;\; \hat{m}_i \in \hat{\Lambda}_W/\hat{\Lambda}_R, \tag{3.10}$$

and

$$N = Order(C \simeq \hat{\Lambda}_W/\hat{\Lambda}_R). \tag{3.11}$$

The S-duality conjecture is:

$$F[\mathbf{e},\hat{\mathbf{m}},1/S,\mathcal{G}]=F[\hat{\mathbf{m}},-\mathbf{e},S,\hat{\mathcal{G}}], \tag{3.12}$$

$$F[\mathbf{e},\hat{\mathbf{m}},S+i,\mathcal{G}]=F[\mathbf{e}+\hat{\mathbf{m}},\hat{\mathbf{m}},S,\mathcal{G}]. \tag{3.13}$$

The transformations $S\to 1/S$ ($\alpha\to 1/\alpha$ for $\theta=0$, namely, "strong-weak coupling duality"), and $S\to S+i$ ($\theta\to\theta+2\pi$) generate the S-duality group, isomorphic to $SL(2,Z)$, and acting on iS by:

$$iS\to\frac{a(iS)+b}{c(iS)+d},\quad a,b,c,d\in Z,\ ad-bc=1. \tag{3.14}$$

These imply, in particular, that (for simply-laced G):

$$\mathcal{Z}(1/S,G)=\mathcal{Z}(S,\hat{G}), \tag{3.15}$$

where

$$\mathcal{Z}(S,G)=\sum_{\mathbf{e}\in(\Lambda_W(G)/\Lambda_R(G))^3,\ \hat{\mathbf{m}}\in(\hat{\Lambda}_W(G)/\hat{\Lambda}_R(G))^3} e^{-\beta F[\mathbf{e},\hat{\mathbf{m}},S]} \tag{3.16}$$

(recall that for $G=\tilde{G}/K, K\subseteq C$, $\Lambda_W(G)=\Lambda_W/K$).

4 The Free Energy and its Properties

In the functional integral representation (3.3), the integration over the scalar zero modes, i.e., the VEVs in the Cartan subalgebra, is divergent. The exact computation of the leading term of such infrared-divergent W, $w[\hat{\mathbf{k}},\hat{\mathbf{m}}]$, is presented in detail in ref. [2]. Here we shall only give the result:

$$w[\hat{\mathbf{k}},\hat{\mathbf{m}}]=K[S]\sum_{\hat{w}_{\mu\nu}}\exp\left[-\pi\sum_{\mu\nu}\left(\beta V\frac{\mathrm{Re}\,S}{2}\frac{(\hat{w}_{\mu\nu}\cdot\hat{w}_{\mu\nu})}{a_\mu^2a_\nu^2}-i\frac{\mathrm{Im}\,S}{4}\epsilon^{\mu\nu\rho\sigma}(\hat{w}_{\mu\nu}\cdot\hat{w}_{\rho\sigma})\right)\right], \tag{4.1}$$

where

$$\hat{w}_{ij}=\epsilon_{ijk}(\hat{l}_k+\hat{m}_k),\qquad \hat{w}_{4i}=\hat{n}_i+\hat{k}_i,\qquad \hat{k}_i,\hat{m}_i\in\hat{\Lambda}_W/\hat{\Lambda}_R,\ \ \hat{l}_i,\hat{n}_i\in\hat{\Lambda}_R. \tag{4.2}$$

A convenient choice for the normalization constant $K[S]$ is:

$$K[S]=\left(\frac{V(\mathrm{Re}\,S)^3}{\beta^3}\right)^{r/2},\quad r=\mathrm{rank}\,G. \tag{4.3}$$

After some algebra, one finds from the twisted functional integrals the free energies:

$$\exp\{-\beta F[\mathbf{e},\hat{\mathbf{m}},S]\}=c\prod_{i=1}^{3}\sum_{k_i\in\Lambda_R,\hat{l}_i\in\hat{\Lambda}_R}\exp\Big\{-\pi\beta_i(k_i+e_i,\hat{l}_i+\hat{m}_i)M(S)\begin{pmatrix}k_i+e_i\\ \hat{l}_i+\hat{m}_i\end{pmatrix}\Big\}, \tag{4.4}$$

where c is a constant, independent of the fluxes $\mathbf{e}$ and $\hat{\mathbf{m}}$ and independent of S, and

$$\beta_i=\frac{\beta a_i^2}{V},\qquad M(S)=\frac{1}{\mathrm{Re}\,S}\begin{pmatrix}1 & \mathrm{Im}\,S\\ \mathrm{Im}\,S & |S|^2\end{pmatrix}=\alpha\begin{pmatrix}1 & a\\ a & \alpha^{-2}+a^2\end{pmatrix}. \tag{4.5}$$

If $\mathcal{G}$ is simply-laced:

$$\exp\{-\beta F[E,M,S,\text{ simply}-\text{laced }\mathcal{G}]\} = c\prod_{i=1}^{3}\sum_{K_i^n,L_i^n=-\infty}^{\infty}\exp\Big\{$$
$$-\pi\beta_i\Big(K_i^n+(E_iC^{-1})^n, L_i^n+(M_iC^{-1})^n\Big)C_{nm}\otimes M(S)\left(\begin{array}{c} K_i^m+(C^{-1}E_i)^m \\ L_i^m+(C^{-1}M_i)^m\end{array}\right)\Big\}, \quad (4.6)$$

where $E_i^n, M_i^n \in Z$, and C_{nm} is the Cartan matrix. It is remarkable that eq. (4.6) is formally equal to the classical piece of a twisted genus-1 string partition function on a toroidal background; the genus-1 modular parameter is S, the target-space background matrix is $C\otimes I_{3\times 3}$, and the twist is (E_i, M_i).[7]

The free energy obeys factorization, Witten's phenomenon and the 't Hooft duality.

- *Factorization*: at $\theta = 0$:

$$F[\mathbf{e},\hat{\mathbf{m}},g,\theta=0] = F[\mathbf{e},0]+F[0,\hat{\mathbf{m}}]+c, \quad (4.7)$$

  where c is independent of the fluxes $\mathbf{e}$ and $\hat{\mathbf{m}}$. Factorization is expected to hold in the limit $a_i,\beta\to\infty$, if we assume that the fluxes occupy only a negligible portion of the total space [16], or if they do not interact, as in the Coulomb phase. The leading IR-divergent contribution to F is scale invariant: $F[La_i, L\beta] = F[a_i,\beta]$ and, therefore, factorization in a large box implies factorization in any box.

- *Witten's Phenomenon*: the free energy for $\theta\neq 0$ is derived from the free energy at $\theta=0$ by the shift:

$$e_i \to e_i + \frac{\theta}{2\pi}\hat{m}_i, \qquad k_i \to k_i + \frac{\theta}{2\pi}\hat{l}_i, \quad (4.8)$$

  Explicitly:

$$\exp\{-\beta F[\mathbf{e},\hat{\mathbf{m}},S]\} = c\prod_{i=1}^{3}\sum_{k_i\in\Lambda_R,\hat{l}_i\in\hat{\Lambda}_R}\exp\Big\{$$
$$-\pi\beta_i(k_i+e_i+\frac{\theta}{2\pi}(\hat{l}_i+\hat{m}_i),\hat{l}_i+\hat{m}_i)M(g)\left(\begin{array}{c} k_i+e_i+\frac{\theta}{2\pi}(\hat{l}_i+\hat{m}_i) \\ \hat{l}_i+\hat{m}_i\end{array}\right)\Big\}, \quad (4.9)$$

  where

$$M(g) = M(S)|_{\theta=0} = \left(\begin{array}{cc} \frac{g^2}{4\pi} & 0 \\ 0 & \frac{4\pi}{g^2}\end{array}\right). \quad (4.10)$$

  This is the Witten phenomenon [20]. Witten's phenomenon also implies that under $\theta\to\theta+2\pi$, the free energy of electric flux $\mathbf{e}$ should transform into the free energy of electric flux $\mathbf{e}+\hat{\mathbf{m}}$. For consistency, one can check that

$$e_i+\hat{m}_i\in\Lambda_W, \qquad k_i+\hat{l}_i\in\Lambda_R. \quad (4.11)$$

[7] This result could be related to the results reported recently in [19].

- *The 't Hooft Duality*: invariance of $W[\hat{\mathbf{k}},\hat{\mathbf{m}}]$ under a discrete $O(4)$ rotation: $1 \leftrightarrow 2, 3 \leftrightarrow 4$, $m_{1,2} \leftrightarrow k_{1,2}$, implies

$$\exp\{-\beta F[e_1,e_2,e_3,\hat{m}_1,\hat{m}_2,\hat{m}_3;a_1,a_2,a_3,\beta]\} = \frac{1}{N^2}\sum_{\hat{k}_1,\hat{k}_2\in\hat{\Lambda}_W/\hat{\Lambda}_R,\, l_1,l_2\in\Lambda_W/\Lambda_R}$$
$$\exp\{2\pi i(\hat{k}_1\cdot e_1+\hat{k}_2\cdot e_2 - l_1\cdot\hat{m}_1 - l_2\cdot\hat{m}_2)\}\exp\{-a_3F[l_1,l_2,e_3,\hat{k}_1,\hat{k}_2,\hat{m}_3,a_2,a_1,\beta,a_3]\}. \tag{4.12}$$

This is the 't Hooft duality relation [16]. Obviously, here there is nothing to prove since we have computed the functional integrals w and, therefore, 't Hooft's duality is automatic.

5 S-Duality

The S-duality transformations of the free energies are:

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2,Z) \quad : \quad & \exp\{-\beta F[\mathbf{e},\hat{\mathbf{m}},S]\}\to \\ & \exp\{-\beta F[\mathbf{e},\hat{\mathbf{m}},(a(iS)+b)/i(c(iS)+d)]\} \\ = \quad & c\prod_{i=1}^{3}\sum_{k_i\in\Lambda_R,\hat{l}_i\in\hat{\Lambda}_R}\exp\left\{-\pi\beta_i(k_i+e_i,\hat{l}_i+\hat{m}_i)AM(S)A^t\begin{pmatrix} k_i+e_i \\ \hat{l}_i+\hat{m}_i\end{pmatrix}\right\} \\ = \quad & \exp\{-\beta F[d\mathbf{e}-b\hat{\mathbf{m}},a\hat{\mathbf{m}}-c\mathbf{e},S]\},\end{aligned} \tag{5.1}$$

where

$$A = \begin{pmatrix} d & -c \\ -b & a\end{pmatrix}. \tag{5.2}$$

In particular,

$$\begin{aligned}\mathcal{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} : S \quad &\to \quad \frac{1}{S}, \qquad F[\mathbf{e},\hat{\mathbf{m}},S]\to \\ F[\mathbf{e},\hat{\mathbf{m}},1/S] \quad &= \quad F[\hat{\mathbf{m}},-\mathbf{e},S],\end{aligned} \tag{5.3}$$

i.e., $\mathcal{S}:\mathcal{G}\to\hat{\mathcal{G}}$ together with $\mathbf{e}\to\mathbf{m},\hat{\mathbf{m}}\to-\hat{\mathbf{e}}$.

$$\begin{aligned}\mathcal{T} = \begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix} : S \quad &\to \quad S+i, \qquad F[\mathbf{e},\hat{\mathbf{m}},S]\to \\ F[\mathbf{e},\hat{\mathbf{m}},S+i] \quad &= \quad F[\mathbf{e}+\hat{\mathbf{m}},\hat{\mathbf{m}},S],\end{aligned} \tag{5.4}$$

i.e., $\mathcal{T}:\mathbf{e}\to\mathbf{e}+\hat{\mathbf{m}}$.

For $\mathcal{G}$ simply-laced, $\hat{\mathcal{G}}=\mathcal{G}$ and, therefore,

$$F\left[\mathbf{e},\mathbf{m},\frac{1}{i}\frac{a(iS)+b}{c(iS)+d}\right] = F[d\mathbf{e}-b\mathbf{m},a\mathbf{m}-c\mathbf{e},S], \tag{5.5}$$

i.e., F is $SL(2,Z)$ covariant.

For $\mathcal{G}$ non-simply-laced, $\mathcal{G} = so(2n+1) \leftrightarrow \hat{\mathcal{G}} = sp(2n)$, and there exist $SL(2,Z)$ transformations that are not allowed (they transform physical fluxes to unphysical ones). For example,

$$\mathcal{TS} : F[\mathbf{e}, \hat{\mathbf{m}}, S] \to F[\hat{\mathbf{m}} - \mathbf{e}, -\mathbf{e}, S]. \tag{5.6}$$

But $\hat{\mathbf{m}} - \mathbf{e}$ is not a vector in $\hat{\Lambda}^3_W$ and, therefore, it is an "illegal" electric flux in $\hat{\mathcal{G}}$.

6 Summary and Remarks

We have defined some gauge invariant quantities in $N=4$ supersymmetric YM theories based on arbitrary compact, simple groups (the generalization to arbitrary compact groups is straightforward): the functional integrals in a box with twisted boundary conditions, $W[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$, and the corresponding free energies, $F[\mathbf{e}, \hat{\mathbf{m}}, S]$. $W[\hat{\mathbf{k}}, \hat{\mathbf{m}}]$ is IR-divergent, and its leading IR-divergent term is exactly computable. Therefore, the corresponding leading term of the free energies in all flux sectors can be derived.

We defined the transformation laws under S-duality of the free energies (the generalization of the MO conjecture to S-duality in the presence of non-Abelian fluxes), and we verified that these laws are obeyed by the quantities we computed. For simply-laced $\mathcal{G}$, $SL(2,Z)$ acts covariantly, but for *non*-simply-laced $\mathcal{G}$, there exist $SL(2,Z)$ transformations that transform physical fluxes into unphysical ones. Therefore, when S is promoted to a true dynamical field, $SL(2,Z)$ is not a true symmetry (but only a sub-group) if G is non-simply-laced. Such gauge groups can never be obtained from $N=4$ toroidal compactifications of the heterotic string and, therefore, in the moduli space of $N=4$ toroidal compactifications, $SL(2,Z)$ S-duality is expected to be a symmetry[8].

Now, it is time to discuss the partition function and electric-magnetic duality. For a gauge *group*, G, not all flux sectors are permitted, but only:

$$e_i \in \frac{\Lambda_W(G)}{\Lambda_R(G)}, \qquad \hat{m}_i \in \frac{\hat{\Lambda}_W(G)}{\hat{\Lambda}_R(G)}. \tag{6.1}$$

Recall that $\Lambda_W(G) = \Lambda_W/K$ for $G = \tilde{G}/K$, $K \subseteq C$ (see section 3 for the other notations). As mentioned before, the partition function, $\mathcal{Z}(S,G)$, is given by summing over all allowed flux sectors:

$$\mathcal{Z}(S,G) = \sum_{\mathbf{e} \in (\Lambda_W(G)/\Lambda_R(G))^3,\ \hat{\mathbf{m}} \in (\hat{\Lambda}_W(G)/\hat{\Lambda}_R(G))^3} e^{-\beta F[\mathbf{e}, \hat{\mathbf{m}}, S]}. \tag{6.2}$$

The electric-magnetic duality is

$$\mathcal{S} \quad : \quad \mathcal{Z}(S,G) \to \mathcal{Z}(1/S,G) = \mathcal{Z}(S,\hat{G}), \qquad \text{if } G \text{ simply} - \text{laced}$$

[8]If different $N=4$ heterotic string backgounds, which are not equivalent to toroidal compactifications, and admit non-simply-laced gauge groups exist, as claimed in ref. [17], only a subgroup of $SL(2,Z)$, described in [2], is expected to be a symmetry in the moduli space of such backgrounds.

$$\mathcal{S} \quad : \quad \mathcal{Z}(S,G) \to \mathcal{Z}(1/S,G) = \mathcal{Z}(S/2,\hat{G}), \qquad \text{if } G \text{ non} - \text{simply} - \text{laced}. \tag{6.3}$$

The partition function $\mathcal{Z}$ is invariant under the subgroup of $SL(2,Z)$ generated by $\{\mathcal{T}^n, \mathcal{S}\mathcal{T}^{\hat{n}}\mathcal{S}\}$, where $n \in Z$ such that $e_i + n\hat{m}_i \in \Lambda_W(G)$ for any $e_i \in \Lambda_W(G)/\Lambda_R(G), \hat{m}_i \in \hat{\Lambda}_W(G)/\hat{\Lambda}_R(G)$, and $\hat{n} \in Z$ such that $\hat{m}_i + \hat{n}e_i \in \hat{\Lambda}_W(G)$ for any $e_i \in \Lambda_W(G)/\Lambda_R(G), \hat{m}_i \in \hat{\Lambda}_W(G)/\hat{\Lambda}_R(G)$.

For example, if $\mathcal{G} = su(2)$:

$$\mathcal{Z}(SU(2)) = \sum_{e_i=0,1/\sqrt{2}} e^{-\beta F[\mathbf{e},0]} = \frac{1}{4}W[0,0]. \tag{6.4}$$

It is invariant under the subgroup $\Gamma_0(2)$ generated by $\{\mathcal{T}, \mathcal{S}\mathcal{T}^2\mathcal{S}\}$. The partition function of the dual group $\hat{SU}(2) = SO(3) \simeq SU(2)/Z_2$ is

$$\mathcal{Z}(SO(3)) = \sum_{m_i=0,1/\sqrt{2}} e^{-\beta F[0,\mathbf{m}]} = \frac{1}{8} \sum_{k_i,m_i=0,1/\sqrt{2}} W[\mathbf{k},\mathbf{m}]. \tag{6.5}$$

Under electric-magnetic duality, indeed,

$$\mathcal{S} : \mathcal{Z}(SU(2)) \leftrightarrow \mathcal{Z}(SO(3)). \tag{6.6}$$

Back to the general case, we should remark that in the Hamiltonian formalism we can evaluate $F[\mathbf{e},\hat{\mathbf{m}}]$ at $\theta = \hat{\mathbf{m}} = 0$, $g \ll 1$. Then, by imposing factorization, the 't Hooft duality and the Witten phenomenon, one can *derive* the result for $\theta, \hat{\mathbf{m}} \neq 0$ from the $\theta = \hat{\mathbf{m}} = 0$ one. The constant scalar fields and gauge fields modes in the CSA, (ϕ_{IJ}, c_i), live on the orbifold [21]:

$$(\phi_{IJ}, c_i) \in \frac{R^{6r} \times \prod_{i=1}^{3} T_i^r}{\text{Weyl} - \text{Group}}, \qquad T_i^r \equiv \frac{R^r}{2\pi\hat{\Lambda}_R/a_i}, \qquad r = \text{rank}\, G. \tag{6.7}$$

More generally, Imbimbo and Mukhi showed how to take into account the scalar divergence in the Hamiltonian approach.

To conclude, we remark that further highly non-trivial tests of S-duality in $N = 4$ supersymmetric YM theories could be done by computing the subleading terms in the IR divergence expansion. Moreover, it would be interesting to see if similar tests can be applied also to $N < 4$ supersymmetric gauge theories.

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