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Infrared-Regulated String Theory and Loop Corrections to Coupling Constants *

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ABSTRACT

Exact superstring solutions are constructed in 4-D space-time, with positive curvature and non-trivial dilaton and antisymmetric tensor fields. The full spectrum of string excitations is derived as a function of moduli fields T^i and the scale $\mu^2 = 1/(k+2)$ which is induced by the non-zero background fields. The spectrum of string excitations has a non-zero mass gap μ^2 and in the weak curvature limit (μ small), μ^2 plays the role of a well defined infrared regulator, consistent with modular invariance, gauge invariance, supersymmetry and chirality.

The effects of a covariantly constant (chomo)magnetic field H, as well as additional curvature can be derived exactly up to one string-loop level. Thus, the one-loop corrections to all couplings (gravitational, gauge and Yukawas) are unambiguously computed and are finite both in the UltraViolet and the InfraRed regime. These corrections are necessary for quantitative string superunification predictions at low energies. The one-loop corrections to the couplings are also found to satisfy Infrared Flow Equations.

Having in our disposal an exact description which goes beyond the leading order in the α'-expansion or the linearized approximation in the background fields, we find interesting clues about the physics of string theory in strong gravitational and magnetic fields. In particular, the nature of gravitational or magnetic instabilities is studied.

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1 Introduction

The low energy properties of the four dimensional $N=1$ superstrings [1]-[7] are described by a special class of $N=1$ supergravity theories [8]-[11], in which all interactions are unified. A sub-class of them seems to extend successfully the validity of the supersymmetric standard model up to the string scale, $M_{str} \sim \mathcal{O}(10^{17})$ GeV. At this energy scale, however, the first excited string states become important and thus the whole effective low energy field theory picture breaks down [12]-[15]. The string unification does not include only the gauge interactions, as it happens in conventional grand unified theories, but also the Yukawa interactions as well as the interactions among the scalars. At energies of order of M_{str}, string unification relations look similar to those of a conventional supersymmetric grand-unified field theory (susy-GUT),

$$\frac{k_i}{\alpha_i(M_{str})} = \frac{1}{\alpha_{str}} \tag{1.1}$$

In susy-GUTs, the normalization constants k_i are fixed *only* for the gauge couplings ($k_1 = k_2 = k_3 = 1$, $k_{em} = \frac{3}{8}$), but there are no relations among gauge and Yukawa couplings at all. In string effective theories, however, the normalization constants (k_i) are known for both gauge and Yukawa interactions. The above unification relations among the couplings are corrected at the quantum level not only by the conventional field theory renormalizable interactions involving light-mass states but also by string corrections involving both the light and the infinite tower of string massive states. It is of main importance that the superstring corrections are finite in the ultraviolet regime and thus one expects to obtain all quantum corrections without ambiguities. In particular one expects the string unification predictions at a scale $\mu < M_{str}$ to have the following form,

$$\frac{k_i}{\alpha_i(\mu)} = \frac{1}{\alpha_{str}} + \frac{b_i}{4\pi}\log\frac{\mu^2}{M_{str}^2} + \Delta_i(T^A). \tag{1.2}$$

The logarithmic behavior in the above formula is due to the light states with masses lower than the scale μ and $\Delta_i(T^A)$ are calculable, *finite* quantities for any particular string solution. Thus, the predictability of a given string solution is extended to all low energy coupling constants once the string-induced corrections are determined. β-functions in string theory were calculated first in [17]

It turns out that $\Delta_i(T^A)$ are non-trivial functions of the vacuum expectation values of some gauge singlet fields [9, 10, 11, 16], the so-called moduli*. The $\Delta_i(T^A)$ are target space duality-invariant functions, which depend on the particular string ground state. Several results for $\Delta_i(T^A)$ exist by now [9, 10, 11, 16] in the exact supersymmetric limit, in many string solutions based on orbifold [2] and fermionic constructions [5]. As we will see later, $\Delta_i(T^A)$ are in principle well defined calculable quantities once we perform our calculations at the string level where all interactions including gravity are consistently defined.

*The moduli fields are flat directions at the string classical level and they remain flat in string perturbation theory, in the exact supersymmetry limit.

Although at this stage we do not know the details of supersymmetry breaking, we should stress here that the corrections to dimensionless coupling constants (e.g. the coefficients of dimension four operators) are still exact if the low energy scale μ is chosen above the threshold of supersymmetric partners of light states, ($m_{susy} \leq \mu$). This is due to the fact that the soft breaking terms in the effective theory do not affect the renormalizations properties of the dimensionless couplings. For the corrections and the structure of the soft breaking parameters only qualitative results exist up to now although this is a subject of an intensive study.

In the past, there was an obstruction in determining the exact form of the string corrections $\Delta_i(T^A)$ due to the infrared divergences of the on-shell calculations in string theory. In a second quantized field theory, we can avoid the IR-divergences due to the massless particles using off-shell calculations. In string theory we cannot do this since string field theory is not very useful computationally up to now and in the first quantized formulation, which is available at present, we do not know how to go consistently off-shell. Even in field theory there are problems in defining an infrared regulator for chiral fermions especially in the presence of space-time supersymmetry.

The idea we will employ here is to modify slightly the ground state of interest in string theory so that it develops a mass gap. It is known already in field theory that a space of negative curvature provides fields (scalars, fermions vectors etc.) with such a mass gap. We will see however that string theory contains the fields (namely the antisymmetric tensor) which when they acquire some suitable expectation values they can provide a mass gap for chiral fermions without running into trouble with anomalies.

Let us indicate here how an expectation value for the dilaton can give masses to bosonic fields. The dilaton couples generically to (massless) bosonic fields in a universal fashion:

$$S_T = \int e^{-2\Phi} \partial_\mu T \partial^\mu T \tag{1.3}$$

where we considered the case of a scalar field T. To find the spectrum of the fluctuations of T we have to define $\tilde{T} = e^{-\Phi} T$ so that kinetic terms are diagonalized. Then, the action becomes

$$S_T = \int \partial_\mu \tilde{T} \partial^\mu \tilde{T} + [\partial_\mu \Phi \partial^\mu \Phi - \partial_\mu \partial^\mu \Phi]\, \tilde{T}^2 \tag{1.4}$$

It is obvious that if $\langle \Phi \rangle = Q_\mu x^\mu$ then the scalar $\tilde{T}$ acquires a mass^2 $Q_\mu Q^{\mu}$† which is positive when Q_μ is spacelike. Similar remarks apply to higher spin bosonic fields. This mechanism via the dilaton cannot give masses to fermions since the extra shift obtained by the diagonalization is a total divergence.

Consider a chiral fermion with its universal coupling to the antisymmetric tensor:

$$S_\psi = \int \bar{\psi} [\overleftrightarrow{\not{\partial}} + \not{H}] \psi \tag{1.5}$$

where $H_\mu = \epsilon_\mu{}^{\nu\rho\sigma} H_{\nu\rho\sigma}$ is the dual of the field strength of the antisymmetric tensor. If $\langle H_\mu \rangle = Q_\mu$, then the Dirac operator acquires a mass gap proportional to $Q_\mu Q^\mu$.

† This was observed in [18] with Q_μ being timelike.

Thus we need to find exact string ground states (CFTs) which implement the mechanism sketched above.

In particular we would like our background to have the following properties:

1. The string spectrum must have a mass gap μ^2. In particular, chiral fermions should be regulated consistently.

2. We should be able to take the limit $\mu^2 \to 0$.

3. It should have $c = (6, 4)$ (in the heterotic case) so that it can be coupled to any internal CFT with $c = (9, 22)$.

4. It should preserve as many spacetime supersymmetries of the original theory, as possible.

5. We should be able to calculate the regulated quantities relevant for the effective field theory.

6. Vertices for spacetime fields (like $F^a_{\mu\nu}$) should be well defined operators on the world-sheet.

7. The theory should be modular invariant (which guarantees the absence of anomalies).

8. Such a regularization should be possible also at the effective field theory level. In this way, calculations in the fundamental theory can be matched without any ambiguity to those of the effective field theory.

CFTs with the properties above employ special four-dimensional spaces as superconformal building blocks with $\hat{c} = 4$ and $N = 4$ superconformal symmetry [12, 14]. The full spectrum of string excitations for the superstring solutions based on those four-dimensional subspaces, can be derived using the techniques developed in [14]. The spectrum does have a mass gap, which is proportional to the curvature of the non-trivial four-dimensional spacetime. Comparing the spectrum in a flat background with that in curved space, we observe a shifting of all massless states by an amount proportional to the spacetime curvature, $\Delta m^2 = Q^2 = \mu^2$, where Q is the Liouville background charge and μ is the IR cutoff. In particular, all gauge symmetries of the original vacuum are spontaneously broken[‡]. What is also interesting is that the shifted spectrum in curved space is equal for bosons and fermions due to the existence of a new space-time supersymmetry defined in curved spacetime [12, 14]. Therefore, our curved spacetime infrared regularization is consistent with supersymmetry, and can be used either in field theory or string theory.

Once we regulate the IR, the one-loop corrections to the couplings can be calculated using the background field method. We will turn on, (chromo)magnetic fields as well as curvature and calculate their effective action at one-loop from which the coupling corrections can be read directly. Of course, since we work in the first quantized formulation the background gauge and gravitational fields have to satisfy the string equations of motion. It turns out that in the IR regulated string theory there are marginal perturbations which

[‡]This is not the usual Higgs type of breaking. Gauge symmetry is spontaneously broken here by non-trivial expectation values of fields in the gravitational sector.

turn on precisely the background fields we need. This provides a new class of exact magnetic field solutions to closed strings[§]. As a byproduct we obtain the exact spectrum of heterotic strings in the presence of such (chromo)magnetic fields.

Finite magnetic fields generically break the spacetime supersymmetries[¶]. We will analyze the presence of tachyonic instabilities induced by such magnetic fields. First, we find [19] that unlike the field theory case, we have a maximum value for the magnetic field

$$H_{\max} = \frac{M^2_{\text{string}}}{\sqrt{2}} \tag{1.6}$$

where, as usual, $M^2_{\text{string}} = 1/\alpha'$. When $H \to H_{\max}$, all states that couple to the magnetic fields (that is, they are either charged or have angular momentum) become infinitely massive and decouple from the theory. This signals the presence of a boundary in the moduli space of the magnetic field.

In field theory there is a critical magnetic field

$$H^{\text{field theory}}_{\text{crit}} \sim \mathcal{O}(\mu^2) \tag{1.7}$$

beyond which the theory is unstable. Here μ is the mass gap of the theory in the charged sector. In the string case there is also a lower critical magnetic field beyond which the theory is unstable but it scales differently

$$H^{\text{string theory}}_{\text{lower crit}} \sim \mathcal{O}(\mu M_{\text{string}}) \tag{1.8}$$

where again μ is the mass gap. This difference is due to the different breaking of gauge symmetry. In our string solutions the gauge symmetries are broken by expectation values of graviton, antisymmetric tensor and dilaton rather than Higgs fields.

In string theory the spectrum is a non-linear function of the magnetic field due to the gravitational backreaction. The effect of the non-linearity is that there is also an upper critical magnetic field $H^{\text{crit}}_{\text{upper}}$ such that $H_{\max} - H^{\text{crit}}_{\text{upper}} \sim \mathcal{O}(\mu M_{str})$ so that the theory is stable for

$$H^{\text{crit}}_{\text{upper}} \leq H \leq H_{\max} \tag{1.9}$$

Similar remarks apply to curvature perturbations. Again, there are tachyonic instabilities due to the breaking of spacetime supersymmetry for a region of the parameters.

Most of the work presented here has already appeared in [19, 20] We present also some new results in section 5.

2 The IR regulated String Theory

We will choose the 4-D CFT which will replace flat space to correspond to the W-space described by the $SU(2)_k \otimes R_Q$ model. It contains a non-compact direction with a linear

[§]Electromagnetic backgrounds in open strings have been discussed in [21]. Magnetic backgrounds in closed string theory have been discussed in [22, 23].

[¶] Internal magnetic fields with emphasis on supersymmetry breaking were discussed recently in [24]. Also the stringy Scherk-Schwarz type of breaking, [25] has a similar interpretation.

dilaton $\Phi = Qx^0$ as well as the $SU(2)_k$ WZW model. Q is related to k as $Q = 1/\sqrt{k+2}$ so that the CFT has the same central charge as flat space. We will define $\mu^2 = 1/(k+2)$, μ is directly related to the mass gap of the regulated theory. The GSO projection couples the SU(2) spin with the spacetime helicity [20]. This effectively projects out the half-integral spins and replaces $SU(2)$ with $SO(3)$. k should be an even positive integer for consistency. For any ground state of the heterotic string with $N < 4$ spacetime supersymmetry the regulated vacuum amplitude turns out to be

$$Z(\mu) = \frac{1}{V(\mu)}\Gamma_0(\mu) Z_0 \tag{2.1}$$

where $V(\mu) = 1/8\pi\mu^3$ is the volume of the nontrivial background and Z_0 is the vacuum amplitude for the unregulated theory, which can be written as

$$Z_0(\tau,\bar{\tau}) = \frac{1}{\mathrm{Im}\tau|\eta|^4}\sum_{a,b=0}^{1}\frac{\theta[^a_b]}{\eta}C[^a_b](\tau,\bar{\tau}) \tag{2.2}$$

where we have separated the generic 4-d contribution. The factor $C[^a_b]$ is the trace in the (^a_b) sector of the internal CFT. Finally, $\Gamma_0(\mu)$ is proportional to the $SO(3)_{k/2}$ partition function

$$\Gamma_0(\mu) = \frac{1}{2}[(\mathrm{Im}\tau)^{\frac{1}{2}}\eta(\tau)\bar{\eta}(\bar{\tau})]^3 \quad \sum_{a,b=0}^{1} e^{-i\pi kab/2}\sum_{l=0}^{k} e^{i\pi bl}\chi_l(\tau)\bar{\chi}_{(1-2a)l+ak}(\bar{\tau}) \tag{2.3}$$

where χ_l are the standard $SU(2)_k$ characters. We have also the correct limit $Z(\mu) \to Z_0$ as $\mu \to 0$.

There is a simple expression for $\Gamma_0(\mu)$

$$\Gamma_0(\mu) = -\frac{1}{2\pi}X'(\mu) \tag{2.4}$$

where prime stands for derivative with respect to μ^2 and

$$X(\mu) = \frac{1}{2\mu}\sum_{m,n\in Z} e^{i\pi(m+n+mn)}\ \exp\left[-\frac{\pi}{4\mu^2\mathrm{Im}\tau}|m-n\tau|^2\right] = \sqrt{\mathrm{Im}\tau}\sum_{m,n\in Z} e^{i\pi n}q^{\frac{1}{4}Q_L^2}\bar{q}^{\frac{1}{4}Q_R^2} \tag{2.5}$$

with

$$Q_L = 2\mu\left(m - \frac{n+1}{2}\right) + \frac{n}{2\mu}\ ,\ \ Q_R = 2\mu\left(m - \frac{n+1}{2}\right) - \frac{n}{2\mu} \tag{2.6}$$

It can be also written in terms of the usual torroidal sum [20]:

$$X(\mu) = Z_T(\mu) - Z_T(2\mu) \tag{2.7}$$

$$Z_T(\mu) = Z_T(1/\mu) = \sqrt{\mathrm{Im}\tau}\sum_{m,n\in Z} q^{\frac{1}{4}(m\mu+n/\mu)^2}\bar{q}^{\frac{1}{4}(m\mu-n/\mu)^2} \tag{2.8}$$

Note that $X(\mu)$ is modular invariant.

The leading infrared behavior can be read from (2.4), (2.5) to be

$$Z(\mu) \to \sqrt{\mathrm{Im}\tau}e^{-\pi\mathrm{Im}\tau\mu^2} \tag{2.9}$$

as $\mathrm{Im}\tau \to \infty$ that indicates explicitly the presence of the mass gap.

More details on this theory can be found in [19, 20].

3 Non-zero $\mathbf{F^a_{\mu\nu}}$ and $R^{\rho\sigma}_{\mu\nu}$ Background in Superstrings

As mentioned in the introduction, in order to calculate the renormalization of the effective couplings we need to turn on backgrounds for gauge and gravitational fields. Thus, our aim is to define the deformation of the two-dimensional superconformal theory which corresponds to a non-zero field strength $F^a_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ background and find the integrated one-loop partition function $Z(\mu, F, \mathcal{R})$, where F is related to the magnitude of the field strength, $F^2 \sim \langle F^a_{\mu\nu} F^{\mu\nu}_a \rangle$ and $\mathcal{R}$ is that of the curvature, $\langle R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \rangle \sim \mathcal{R}^2$.

$$Z[\mu, F_i, \mathcal{R}] = \frac{1}{V(\mu)} \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{(\mathrm{Im}\tau)^2} Z[\mu, F_i, \mathcal{R}; \tau, \bar{\tau}] \tag{3.1}$$

The index i labels different simple or $U(1)$ factors of the gauge group of the ground state.

In flat space, a small non-zero $F^a_{\mu\nu}$ background gives rise to an infinitesimal deformation of the 2-d σ-model action given by,

$$\Delta S^{2d}_{\mathbf{flat}} = \int dz d\bar{z}\; F^a_{\mu\nu} [x^\mu \partial_z x^\nu + \psi^\mu \psi^\nu] \bar{J}^a \tag{3.2}$$

Observe that for $F^a_{\mu\nu}$ constant (constant magnetic field), the left moving operator $[x^\mu \partial_z x^\nu + \psi^\mu \psi^\nu]$ is not a well-defined $(1,0)$ operator on the world sheet. Even though the right moving Kac-Moody current $\bar{J}_a$ is a well-defined $(0,1)$ operator, the total deformation is not integrable in flat space. Indeed, the 2-d σ-model β-functions are not satisfied in the presence of a constant magnetic field. This follows from the fact that there is a *non-trivial back-reaction* on the gravitational background due the non-zero magnetic field.

In the W-space, however, the vertex operator which turns on a (chromo)magnetic field background B^a_i is

$$V^a_i = (J^i + \frac{1}{2} \epsilon^{i,j,k} \psi^j \psi^k) \bar{J}^a \tag{3.3}$$

This vertex operator is of the current-current type. In order for such perturbations to be marginal (equivalently the background to satisfy the string equations of motion) we need to pick a single index i, which we choose to be $i = 3$ and need to restrict the gauge group index a to be in the Cartan of the gauge group. We will normalize the antiholomorphic currents $\bar{J}^a_i$ in each simple or U(1) component G_i of the gauge group G as

$$\langle \bar{J}^a_i(\bar{z}) \bar{J}^b_j(0) \rangle = \frac{k_i \delta^{ij}}{2} \frac{\delta^{ab}}{\bar{z}^2} \tag{3.4}$$

With this normalization, the field theory gauge coupling is $g_i^2 = 2/k_i$. Thus the most general marginal (chromo)magnetic field is generated from the following vertex operator

$$V_{magn} = \frac{(J^3 + \psi^1 \psi^2)}{\sqrt{k+2}} \frac{\vec{F}_i \cdot \bar{\vec{J}}_i}{\sqrt{k_i}} \tag{3.5}$$

where the index i labels the simple or $U(1)$ components G_i of the gauge group and $\bar{\vec{J}}_i$ is a r_i-dimensional vector of currents in the Cartan of the group G_i (r_i is the rank of G_i). The repeated index i implies summation over the simple components of the gauge group.

We would like to obtain the exact one-loop partition function in the presence of such perturbation. Since this is an abelian current-current perturbation, the deformed partition function can be obtained by an $O(1,N)$, boost ($N=\sum_i r_i$) of the charged lattice of the undeformed partition function, computed in the previous section.

We will indicate the method in the case where we turn on a single magnetic field F, in a gauge group factor with central element k_g, in which case

$$V_F = F\frac{(J^3+\psi^1\psi^2)}{\sqrt{k+2}}\frac{\bar{J}}{\sqrt{k_g}} \tag{3.6}$$

Let us denote by $\mathcal{Q}$ the zero mode of the holomorphic helicity current $\psi^1\psi^2$, $\bar{\mathcal{P}}$ the zero mode of the antiholomorphic current $\bar{J}$ and $I,\bar{I}$ the zero modes of the $SU(2)$ currents $J^3,\bar{J}^3$ respectively. Then, the relevant parts of L_0 and $\bar{L}_0$ are

$$L_0=\frac{\mathcal{Q}^2}{2}+\frac{I^2}{k}+\cdots \quad , \quad \bar{L}_0=\frac{\bar{\mathcal{P}}^2}{k_g}+\cdots \tag{3.7}$$

We will rewrite L_0 as

$$L_0=\frac{(\mathcal{Q}+I)^2}{k+2}+\frac{k}{2(k+2)}\left(\mathcal{Q}-\frac{2}{k}I\right)^2+\cdots \tag{3.8}$$

where we have separated the relevant supersymmetric zero mode $\mathcal{Q}+I$ and its orthogonal complement $\mathcal{Q}-2I/k$ which will be a neutral spectator to the perturbing process. What remains to be done is an $O(1,1)$ boost that mixes the holomorphic current $\mathcal{Q}+I$ and the antiholomorphic one $\bar{\mathcal{P}}$. This is straighforward with the result

$$L_0'=\frac{k}{2(k+2)}\left(\mathcal{Q}-\frac{2}{k}I\right)^2+\left(\cosh x\frac{\mathcal{Q}+I}{\sqrt{k+2}}+\sinh x\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right)^2+\cdots \tag{3.9}$$

$$\bar{L}_0'=\left(\sinh x\frac{\mathcal{Q}+I}{\sqrt{k+2}}+\cosh x\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right)^2+\cdots \tag{3.10}$$

where x is the parameter of the $O(1,1)$ boost. Thus we obtain from (3.9), (3.10) the change of L_0, $\bar{L}_0$ as

$$\delta L_0\equiv L_0'-L_0=\delta\bar{L}_0\equiv\bar{L}_0'-\bar{L}_0=F\frac{(\mathcal{Q}+I)}{\sqrt{k+2}}\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}+\frac{\sqrt{1+F^2}-1}{2}\left[\frac{(\mathcal{Q}+I)^2}{k+2}+\frac{\bar{\mathcal{P}}^2}{k_g}\right] \tag{3.11}$$

where we have identified

$$F\equiv\sinh(2x) \tag{3.12}$$

We are now able to compute with the more general marginal perturbation which is a sum of the general magnetic perturbation (3.5) and the gravitational perturbation

$$V_{grav}=\mathcal{R}\frac{(J^3+\psi^1\psi^2)}{\sqrt{k+2}}\frac{\bar{J}^3}{\sqrt{k}} \tag{3.13}$$

The only extra ingredient we need is an $O(1+N)$ transformation to mix the antiholomorphic currents. Thus, we obtain

$$\delta L_0 = \delta\bar{L}_0 = \left[\frac{\mathcal{R}\bar{I}}{\sqrt{k}} + \frac{\vec{F}_i\cdot\bar{\vec{\mathcal{P}}}_i}{\sqrt{k_i}}\right]\frac{(\mathcal{Q}+I)}{\sqrt{k+2}} + \tag{3.14}$$

$$+\frac{\sqrt{1+\mathcal{R}^2+\vec{F}_i\cdot\vec{F}_i}-1}{2}\left[\frac{(\mathcal{Q}+I)^2}{k+2} + (\mathcal{R}^2+\vec{F}_i\cdot\vec{F}_i)^{-1}\left(\frac{\mathcal{R}\bar{I}}{\sqrt{k}} + \frac{\vec{F}_i\cdot\bar{\vec{\mathcal{P}}}_i}{\sqrt{k_i}}\right)^2\right]$$

From now on we focus in the case where we have a single (chromo)magnetic field F as well as the curvature perturbation $\mathcal{R}$. Then (3.14) simplifies to

$$\delta L_0 = \delta\bar{L}_0 = \left[\mathcal{R}\frac{\bar{I}}{\sqrt{k}} + F\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right]\frac{(\mathcal{Q}+I)}{\sqrt{k+2}} + \tag{3.15}$$

$$+\frac{\sqrt{1+\mathcal{R}^2+F^2}-1}{2}\left[\frac{(\mathcal{Q}+I)^2}{k+2} + (\mathcal{R}^2+F^2)^{-1}\left(\mathcal{R}\frac{\bar{I}}{\sqrt{k}} + F\frac{\bar{\mathcal{P}}}{\sqrt{k_i}}\right)^2\right]$$

Eq. (3.15) can be written in the following form which will be useful in order to compare with the field theory limit

$$\delta L_0 = \frac{1+\sqrt{1+F^2+\mathcal{R}^2}}{2}\left[\frac{(\mathcal{Q}+I)}{\sqrt{k+2}} + \frac{1}{1+\sqrt{1+F^2+\mathcal{R}^2}}\left(\mathcal{R}\frac{\bar{I}}{\sqrt{k}} + F\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right)\right]^2 \tag{3.16}$$

$$-\frac{(\mathcal{Q}+I)^2}{k+2}$$

and for $\mathcal{R}=0$ as

$$\delta L_0 = \frac{1+\sqrt{1+F^2}}{2}\left[\frac{(\mathcal{Q}+I)}{\sqrt{k+2}} + \frac{F}{1+\sqrt{1+F^2}}\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right]^2 - \frac{(\mathcal{Q}+I)^2}{k+2} \tag{3.17}$$

Eq. (2.1) along with (3.14) provide the complete and exact spectrum of string theory in the presence of the (chromo)magnetic fields $\vec{F}_i$ and curvature $\mathcal{R}$. We will analyse first the case of a single magnetic field F and use (3.17). Since for physical states $L_0 = \bar{L}_0$ it is enough to look at $L_0 = M_L^2$ which in our conventions is the side that has $N=1$ superconformal symmetry.

$$M_L^2 = -\frac{1}{2} + \frac{\mathcal{Q}^2}{2} + \frac{1}{2}\sum_{i=1}^{3}\mathcal{Q}_i^2 + \frac{(j+1/2)^2-(\mathcal{Q}+I)^2}{k+2} + E_0 + \tag{3.18}$$

$$\frac{1+\sqrt{1+F^2}}{2}\left[\frac{(\mathcal{Q}+I)}{\sqrt{k+2}} + \frac{F}{1+\sqrt{1+F^2}}\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right]^2$$

where, the $-1/2$ is the universal intercept in the N=1 side, $\mathcal{Q}_i$ are the internal helicity operators (associated to the internal left-moving fermions), E_0 contains the oscillator contributions as well as the internal lattice (or twisted) contributions, and $j = 0, 1, 2, \cdots, k/2^*$, $j \geq |I| \in Z$. We can see already a reason here for the need of the SO(3) projection. We do not want half-integral values of I to change the half-integrality of the spacetime helicity $\mathcal{Q}$.

Let us look first at how the low lying spectrum of space-time fermions is modified. For this we have to take $\mathcal{Q} = \mathcal{Q}_i = \pm 1/2$. Then M_L^2 can be written as a sum of positive factors, $E_0 \geq 0$, $(j+1/2)^2 \geq (\pm 1/2 + I)^2$ and

$$\frac{1+\sqrt{1+F^2}}{2}\left[\frac{(\mathcal{Q}+I)}{\sqrt{k+2}} + \frac{F}{1+\sqrt{1+F^2}}\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right]^2 \geq 0 \tag{3.19}$$

Thus fermions cannot become tachyonic and this a good consistency check for our spectrum since a "tachyonic" fermion is a ghost. This argument can be generalized to all spacetime fermions in the theory.

Bosonic states can become tachyonic though, but for this to happen, as in field theory they need to have non-zero helicity. It can be shown that for k positive only $|\mathcal{Q}| = 1$, $j = |I| = 0, 1, 2, \cdots, k/2$ states can become tachyonic†.

By also imposing $L_0 = \bar{L}_0$ we obtain

$$\mathcal{Q}^2 - \frac{2}{k_g}\bar{\mathcal{P}}^2 + 1 \geq 0 \tag{3.20}$$

and thus the minimal value for M_L^2 can be written as

$$M_{min}^2 = \frac{\mathcal{Q}^2-1}{2} + \frac{(|I|+1/2)^2 - (\mathcal{Q}+I)^2}{k+2} + \frac{1+\sqrt{1+F^2}}{2}\left[\frac{(\mathcal{Q}+I)}{\sqrt{k+2}} + \frac{F}{1+\sqrt{1+F^2}}\frac{\bar{\mathcal{P}}}{\sqrt{k_g}}\right]^2 \tag{3.21}$$

Let us introduce the variables

$$H = \frac{F}{\sqrt{2}(1+\sqrt{1+F^2})} \quad , \quad e = \sqrt{\frac{2}{k_g}}\bar{\mathcal{P}} \tag{3.22}$$

H is the natural magnetic field from the σ-model point of view [19] and e is the charge. Notice that while F varies along the whole real line, $|H| \leq 1/\sqrt{2}$. From (3.20)

$$e^2 \leq \mathcal{Q}^2 + 1 \tag{3.23}$$

Then, there are tachyons provided

$$\frac{1}{1-2H^2}\left(\frac{(\mathcal{Q}+I)}{\sqrt{k+2}} + eH\right)^2 + \frac{\mathcal{Q}^2-1}{2} + \frac{(|I|+1/2)^2-(\mathcal{Q}+I)^2}{k+2} \leq 0 \tag{3.24}$$

*Remember that k is an even integer for $SO(3)$.

†This is unlike the case of [23] where states with higher helicities become tachyonic.

In fact it is not difficult to see that the first instability appears due to $I = 0$ states becoming tachyonic. We will leave the charge free for the moment, although there are certainly constraints on it, depending on the gauge group. For example for the E_6 or E_8 groups we have $e^2_{min} = 1/4$, and for all realistic non-abelian gauge groups $e_{min} = \mathcal{O}(1)$. For torroidal $U(1)$'s however e_{min} can become arbitrarily small by tuning the parameters of the torus. Note however that in any case for the potential tachyonic states with $|\mathcal{Q}| = 1$ the charge must satisfy

$$\frac{1}{2(k+2)} \le e^2 \le 2 \tag{3.25}$$

Thus for $|\mathcal{Q}| = 1$ we obtain the presence of tachyons provided that

$$H^{\rm crit}_{\rm min} \le |H| \le H^{\rm crit}_{\rm max} \tag{3.26}$$

with

$$H^{\rm crit}_{\rm min} = \frac{\mu}{|e|} \frac{1 - \frac{\sqrt{3}}{2}\sqrt{1 - \frac{1}{2}\left(\frac{\mu}{e}\right)^2}}{1 + \frac{3}{2}\left(\frac{\mu}{e}\right)^2} \tag{3.27}$$

$$H^{\rm crit}_{\rm max} = \frac{\mu}{|e|} \frac{J + 1 + \sqrt{\left(J + \frac{3}{4}\right)\left(1 - 2\left(J + \frac{1}{2}\right)^2 \frac{\mu^2}{e^2}\right)}}{1 + \left(2J + \frac{3}{2}\right)\frac{\mu^2}{e^2}} \tag{3.28}$$

where

$$J = \text{integral part of} \quad -\frac{1}{2} + \frac{|e|}{\sqrt{2}\mu} \tag{3.29}$$

We have also introduced the IR cutoff scale $k + 2 = 1/\mu^2$.

We note that for small μ and $|e| \sim \mathcal{O}(1)$ $H^{\rm crit}_{\rm min}$ is of order $\mathcal{O}(\mu)$. However $H^{\rm crit}_{\rm max}$ is below $H_{\rm max} = 1/\sqrt{2}$[‡] by an amount of order $\mathcal{O}(\mu)$. Thus for small values of H there are no tachyons until a critical value $H^{\rm crit}_{\rm min}$ where the theory becomes unstable. For $|H| \ge H^{\rm crit}_{\rm max}$ the theory is stable again till the boundary $H = 1/\sqrt{2}$. It is interesting to note that if there is a charge in the theory with the value $|e| = \sqrt{2}\mu$ then $H^{\rm crit}_{\rm max} = 1/\sqrt{2}$ so there is no region of stability for large magnetic fields. For small μ there are always charges satisfying (3.25) which implies that there is always a magnetic instability. However even for $\mu = \mathcal{O}(1)$ it seems (although we have no rigorous proof) that the magnetic instability is present independent of the nature of the gauge group (provided it has charged states in the perturbative spectrum).

The behavior above should be compared to the field theory behavior

$$E^2 = p_3^2 + M^2 + eH(2n + 1 - gS) \tag{3.30}$$

In (3.30) we have an instability provided there is a particle with $gS \ge 1$. Then the theory is unstable for

$$|H| \ge \frac{M^2}{|e|(gS - 1)} \tag{3.31}$$

[‡]We will frequently use dimensionless notation, $\alpha' = 1$. Dimensions can be easily reinstated.

where M is the mass of the particle (or the mass gap). However there is no restauration of stability for large values of H. This happens in string theory due to the backreaction of gravity. There is also another difference. In field theory, $H_{crit} \sim \mu^2$ while in string theory $H_{crit} \sim \mu M_{str}$ where we denoted by μ the mass gap in both cases and $M_{str}^2 = 1/\alpha'$. We should also note that in a classically gapless theory like unbroken Yang-Mills we obtain that the trivial ground state is unstable even for infinitesimal magnetic fields. This a tree level indication that this is not the correct ground state of the theory, which of course can be verified at one-loop where one learns that the coupling is strong in the IR and the theory probably confines and has a mass gap.

Let us now study the gravitational perturbation. Using (3.15) the mass formula is (in analogy with (3.18)

$$M_L^2 = -\frac{1}{2} + \frac{\mathcal{Q}^2}{2} + \frac{1}{2}\sum_{i=1}^{3}\mathcal{Q}_i^2 + \frac{(j+1/2)^2 - (\mathcal{Q}+I)^2}{k+2} + E_0 + \tag{3.32}$$
$$+\frac{1+\sqrt{1+\mathcal{R}^2}}{2}\left[\frac{(\mathcal{Q}+I)}{\sqrt{k+2}} + \frac{\mathcal{R}}{1+\sqrt{1+\mathcal{R}^2}}\frac{\bar{I}}{\sqrt{k}}\right]^2$$

Introducing the σ-model variable

$$\lambda = \sqrt{\mathcal{R}+\sqrt{1+\mathcal{R}^2}} \quad , \quad \frac{1}{\lambda} = \sqrt{-\mathcal{R}+\sqrt{1+\mathcal{R}^2}} \tag{3.33}$$

(3.32) becomes

$$M_L^2 = -\frac{1}{2} + \frac{\mathcal{Q}^2}{2} + \frac{1}{2}\sum_{i=1}^{3}\mathcal{Q}_i^2 + \frac{(j+1/2)^2 - (\mathcal{Q}+I)^2}{k+2} + E_0 + \tag{3.34}$$
$$+\frac{1}{4}\left[\left(\lambda+\frac{1}{\lambda}\right)\frac{(\mathcal{Q}+I)}{\sqrt{k+2}} + \left(\lambda-\frac{1}{\lambda}\right)\frac{\bar{I}}{\sqrt{k}}\right]^2$$

Only $|\mathcal{Q}| = 1$ and $j = |I| = 0, 1, \cdots, k/2$, can produce tachyonic instabilities. Due to the $\lambda \to 1/\lambda$ duality we will restrict ourselves to the region $\lambda \leq 1$.

Thus, the condition for existence of tachyons is

$$\frac{1}{4}\left[\left(\lambda+\frac{1}{\lambda}\right)\frac{(\mathcal{Q}+I)}{\sqrt{k+2}} + \left(\lambda-\frac{1}{\lambda}\right)\frac{\bar{I}}{\sqrt{k}}\right]^2 + \frac{\mathcal{Q}^2-1}{2} + \frac{(|I|+1/2)^2 - (\mathcal{Q}+I)^2}{k+2} \leq 0 \tag{3.35}$$

Thus the state with quantum numbers $(I, \bar{I})$ becomes tachyonic when

$$\lambda_{min}^2 \leq \lambda^2 \leq \lambda_{max}^2 \tag{3.36}$$

with

$$\lambda_{max}^2 = \frac{\frac{\bar{I}^2}{k} - \frac{I^2-1/2}{k+2} + \sqrt{\frac{(I+3/4)}{k+2}\left(\frac{\bar{I}^2}{k} - \frac{(I+1/2)^2}{k+2}\right)}}{\left(\frac{I}{\sqrt{k+2}} + \frac{\bar{I}}{\sqrt{k}}\right)^2} \tag{3.37}$$

$$\lambda_{min}^2 = \frac{\frac{\bar{I}^2}{k} - \frac{I^2-1/2}{k+2} - \sqrt{\frac{(I+3/4)}{k+2}\left(\frac{\bar{I}^2}{k} - \frac{(I+1/2)^2}{k+2}\right)}}{\left(\frac{I}{\sqrt{k+2}} + \frac{\bar{I}}{\sqrt{k}}\right)^2} \tag{3.38}$$

For large k, λ_{max} approaches one, however at the same time the instability region shrinks to zero so that in the limit $\lambda = 1, k = \infty$ flat space is stable.

4 Trace Formulae for small Magnetic Fields

In this section, we will treat the magnetic fields F_i, as well as the curvature perturbation $\mathcal{R}$ as small, and we will derive trace formulae for averages of polynomials in this parameters, for applications to the evaluation of loop corrections to the appropriate coupling constants.

We will need a single magnetic field F_i for each simple or U(1) factor of the gauge group and a different normalization than the one used in (3.14)

$$F_i \to \sqrt{k_i(k+2)}F_i \quad , \quad \mathcal{R} \to \sqrt{k(k+2)}\mathcal{R} \tag{4.1}$$

Then (3.14) becomes

$$\delta L_0 = \delta \bar{L}_0 = (\mathcal{Q}+I)(\mathcal{R}\bar{I} + F_i\bar{\mathcal{P}}_i) + \tag{4.2}$$
$$+\frac{-1+\sqrt{1+(k+2)(k_iF_i^2+k\mathcal{R}^2)}}{2}\left[\frac{(\mathcal{Q}+I)^2}{k+2} + \frac{(F_i\bar{\mathcal{P}}_i + \mathcal{R}\bar{I})^2}{k_iF_i^2+k\mathcal{R}^2}\right]$$

The first term is the linearized perturbation while the second is the backreaction necessary for conformal and modular invariance. The unperturbed partition function can be written as

$$Tr[\exp[-2\pi \mathrm{Im}\tau(\mathrm{L}_0+\bar{\mathrm{L}}_0) + 2\pi i \mathrm{Re}\tau(\mathrm{L}_0 - \bar{\mathrm{L}}_0)]] \tag{4.3}$$

Expanding the perturbed partition function in a power series in $F_i, \mathcal{R}$

$$Z(\mu, F, \mathcal{R}) = \sum_{n_i,m=0}^{\infty}\prod_{i=1}^{n} F_i^{n_i}\mathcal{R}^m Z_{n_i,m}(\mu) \tag{4.4}$$

we can extract the integrated correlators $Z_{n_i,m} = \langle \prod_{i=1}^n F_i^{n_i} R^m\rangle$ (n is the number of simple components of the gauge group).

$$\langle F_i\rangle = -4\pi \mathrm{Im}\tau \langle(\mathcal{Q}+I)\rangle\langle\bar{\mathcal{P}}_i\rangle \tag{4.5a}$$

$$\langle \mathcal{R}\rangle = -4\pi \mathrm{Im}\tau \langle(\mathcal{Q}+I)\rangle\langle\bar{I}\rangle \tag{4.5b}$$

$$\langle F_i^2\rangle = 8\pi^2 \mathrm{Im}\tau^2\left[\langle(\mathcal{Q}+I)^2\rangle - \frac{(k+2)}{8\pi \mathrm{Im}\tau}\right]\left[\langle(\bar{\mathcal{P}}_i)^2\rangle - \frac{k_i}{8\pi\mathrm{Im}\tau}\right] - \frac{k_i(k+2)}{8} \tag{4.5c}$$

$$\langle \mathcal{R}^2\rangle = 8\pi^2 \mathrm{Im}\tau^2\left[\langle(\mathcal{Q}+I)^2\rangle - \frac{k+2}{8\pi \mathrm{Im}\tau}\right]\left[\langle(\bar{I})^2\rangle - \frac{k}{8\pi\mathrm{Im}\tau}\right] - \frac{k(k+2)}{8} \tag{4.5d}$$

$$\langle \mathcal{R}F_i\rangle = 16\pi^2 \mathrm{Im}\tau^2 \langle\bar{I}\bar{\mathcal{P}}_i\rangle\left[\langle(\mathcal{Q}+I)^2\rangle - \frac{k+2}{8\pi \mathrm{Im}\tau}\right] \tag{4.5e}$$

$$\langle F_iF_j\rangle = 16\pi^2 \mathrm{Im}\tau^2 \langle\bar{\mathcal{P}}_i\bar{\mathcal{P}}_j\rangle\left[\langle(\mathcal{Q}+I)^2\rangle - \frac{k+2}{8\pi \mathrm{Im}\tau}\right] \tag{4.5f}$$

where we should always remember that $k+2 = 1/\mu^2$. We should note here that for torroidal $U(1)$ gauge fields there is another natural basis in which $\langle\bar{J}_i(1)\bar{J}_j(0)\rangle = G_{ij}/2$ where G_{ij} is the constant metric of the torus. Then the trace formula becomes

$$\langle F_iF_j\rangle = 8\pi^2 \mathrm{Im}\tau^2\left[\langle(\mathcal{Q}+I)^2\rangle - \frac{(k+2)}{8\pi \mathrm{Im}\tau}\right]\left[\langle(\bar{\mathcal{P}}_i)^2\rangle - \frac{G_{ij}}{8\pi\mathrm{Im}\tau}\right] - \frac{G_{ij}(k+2)}{8} \tag{4.6}$$

For Supersymmetric ground states we have simplifications

$$\langle F_i^2\rangle_{SUSY} = 8\pi^2 \mathrm{Im}\tau^2 \langle \mathcal{Q}^2\rangle \left[\langle(\bar{\mathcal{P}}_i)^2\rangle - \frac{k_i}{8\pi\mathrm{Im}\tau}\right] \tag{4.7}$$

$$\langle \mathcal{R}^2\rangle_{SUSY} = 8\pi^2 \mathrm{Im}\tau^2 \langle \mathcal{Q}^2\rangle \left[\langle(\bar{I})^2\rangle - \frac{k}{8\pi\mathrm{Im}\tau}\right] \tag{4.8}$$

Renormalizations of higher terms can be easily computed. We give here the expression for an F_i^4 term,

$$\begin{aligned}\langle F_i^4\rangle = \frac{(4\pi\mathrm{Im}\tau)^4}{24}\langle\Big[(\mathcal{Q}+I)^4\bar{\mathcal{P}}_i^4 - \frac{3}{4\pi\mathrm{Im}\tau}(\mathcal{Q}+I)^2\bar{\mathcal{P}}_i^2\Big((k_i(\mathcal{Q}+I)^2 + \\ + (k+2)\bar{\mathcal{P}}_i^2\Big) + \frac{3}{4(4\pi\mathrm{Im}\tau)^2}\Big[k_i(\mathcal{Q}+I)^2 + (k+2)\bar{\mathcal{P}}_i^2\Big]^2 - \\ - \frac{3k_i(k+2)}{2(4\pi\mathrm{Im}\tau)^3}\Big[[k_i(\mathcal{Q}+I)^2 + (k+2)\bar{\mathcal{P}}_i^2]\Big]\rangle\end{aligned} \tag{4.9}$$

The charge $\mathcal{Q}$ in the above formulae acts on the helicity ϑ-function $\vartheta\left[{\alpha \atop \beta}\right](\tau, v)$ as differentiation with respect to v divided by $2\pi i$. The charges $\bar{\mathcal{P}}_i$ act also as v derivatives on the respective characters of the current algebra. $I, \bar{I}$ act on the level-k ϑ-function present in $SO(3)_{k/2}$ partition function (due to the parafermionic decomposition).

5 One-loop Corrections to the Coupling Constants

We now focus on the one-loop correction to the gauge couplings. Bearing anomalous U(1)'s we can immediately see from (4.5) that $\langle F_i\rangle = 0$ and $\langle F_i F_j\rangle = 0$ for $i \neq j$. The conventionally normalized one-loop correction is

$$\frac{16\pi^2}{g_i^2}|_{1-loop} = -\frac{1}{(2\pi)^2}\int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\tau^2}\langle F_i^2\rangle \tag{5.1}$$

Putting everything together we obtain

$$\begin{aligned}\frac{16\pi^2}{g_i^2}|_{1-loop} = -\frac{i}{\pi^2 V(\mu)}\int_{\mathcal{F}}\frac{d^2\tau}{\mathrm{Im}\tau|\eta|^4}\sum_{a,b=0}^{1}\left[X'(\mu)\partial_\tau\left(\frac{\theta[{a \atop b}]}{\eta}\right) + \frac{1}{6\mu^2}\dot{X}'(\mu)\frac{\theta[{a \atop b}]}{\eta}\right] \times \\ \times Tr^I_{a,b}\left[\langle\bar{\mathcal{P}}_i^2\rangle - \frac{k_i}{8\pi\mathrm{Im}\tau}\right] - -\frac{k_i}{64\pi^3\mu^2 V(\mu)}\int_{\mathcal{F}}\frac{d^2\tau}{\mathrm{Im}\tau^2}X'(\mu)Z_0\end{aligned} \tag{5.2}$$

where dot stands for derivative with respect to τ and Tr^I_{ab} stands for the trace in the $({a \atop b})$ sector of the internal CFT. Eq. (5.2) is valid also for non-supersymmetric ground states.

When we have $N \geq 1$ supersymmetry it simplifies to*

$$\frac{16\pi^2}{g_i^2}|^{SUSY}_{1-loop} = -\frac{i}{\pi^2 V(\mu)} \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\tau|\eta|^4} \sum_{a,b=0}^{1} \left[X'(\mu)\frac{\partial_\tau \theta[^a_b]}{\eta}\right] Tr^I_{a,b}\left[\langle \bar{\mathcal{P}}_i^2\rangle - \frac{k_i}{8\pi \mathrm{Im}\tau}\right] \tag{5.3}$$

The general formula (5.2) can be split in the following way

$$\frac{16\pi^2}{g_i^2}|_{1-loop} = I_1 + I_2 + I_3 \tag{5.4}$$

$$I_1 = -\frac{i}{\pi^2 V(\mu)} \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\tau|\eta|^4} X'(\mu) \sum_{a,b=0}^{1} \partial_\tau \left(\frac{\theta[^a_b]}{\eta}\right) Tr^I_{a,b}\left[\langle(\bar{\mathcal{P}}_i^2\rangle - \frac{k_i}{8\pi \mathrm{Im}\tau}\right] \tag{5.5}$$

$$I_2 = -\frac{i}{6\pi^2\mu^2 V(\mu)} \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\tau|\eta|^4} \dot{X}'(\mu) \sum_{a,b}^{1} \frac{\theta[^a_b]}{\eta} Tr^I_{a,b}\left[\langle \bar{\mathcal{P}}_i^2\rangle - \frac{k_i}{8\pi \mathrm{Im}\tau}\right] \tag{5.6}$$

$$I_3 = -\frac{k_i}{64\pi^3\mu^2 V(\mu)} \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\tau^2} X'(\mu) Z_0 \tag{5.7}$$

All the integrands are separately modular invariant. The universal term in I_1 is due to an axion tadpole. I_3 is the contribution of a dilaton tadpole. I_2 are extra helicity contributions due to the curved background. Moreover I_2, I_3 have power IR divergences which reflect quadratic divergences in the effective field theory. I_2, I_3 are zero for supersymmetric ground states due to the vanishing of the sum of the helicity theta functions.

We will now analyse the contribution of the massless sector to the one-loop corrections. Since

$$-\frac{1}{i\pi}\partial_\tau \left(\frac{\theta[^a_b]}{\eta}\right) \to (-1)^F \left(\frac{1}{12} - \chi^2\right) \tag{5.8}$$

where χ is the helicity of a state, we obtain

$$I_1^{massless} = -\frac{1}{\pi} Str\left[\bar{\mathcal{P}}_i^2 \left(\frac{1}{12} - \chi^2\right)\right] \mathbf{J}_1(\mu) + \frac{k_i}{8\pi^2} Str\left[\frac{1}{12} - \chi^2\right] \mathbf{J}_2(\mu) \tag{5.9}$$

$$I_2^{massless} = -\frac{1}{12\pi^2\mu^2} Str[\bar{\mathcal{P}}_i^2]\mathbf{J}_2(\mu) + \frac{k_i}{48\pi^3\mu^2} Str[\mathbf{1}]\mathbf{J}_3(\mu) \tag{5.10}$$

$$I_3^{massless} = -\frac{k_i}{64\pi^3\mu^2} Str[\mathbf{1}]\mathbf{J}_3(\mu) \tag{5.11}$$

Here

$$\mathbf{J}_n \equiv \frac{1}{V(\mu)} \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\tau^n} X'(\mu) \tag{5.12}$$

which can be evaluated to be

$$\mathbf{J}_1(\mu) = 2\pi \log \mu^2 + 2\pi(\log \pi + \gamma_E - 3 + \frac{3}{2}\log 3) + \mathcal{O}(e^{-\frac{1}{\mu^2}}) \tag{5.13}$$

$$\mathbf{J}_2(\mu) = -\frac{4\pi^2}{3}(1+\mu^2) \ , \ \ \mathbf{J}_3(\mu) = -\pi \log 3 - \frac{28\pi^3}{15}\mu^4 + \mathcal{O}(e^{-\frac{1}{\mu^2}}) \tag{5.14}$$

*This formula appeared in [20] in a slightly different notation.

We would like now to describe the same calculation in the effective field theory.

This calculation proceeds along the same lines as above taking into account the following differences.

- Now the mass gap is $\mu_e^2 = 1/k$ and $V(\mu_e) = 1/(8\pi\mu_e^3)$.
- $\Gamma_0/V(\mu_e)$ is given by the momentum mode part of the stringy expression:

$$\frac{\Gamma_0}{V(\mu_e)} = -4\mu_e^3 \partial_{\mu_e^2} \sqrt{\mathrm{Im}\tau} \sum_{n\in Z} e^{-\pi \mathrm{Im}\tau \mu_e^2 (2n+1)^2} \tag{5.15}$$

- There is an incomplete cancelation of the $1/8\pi\mu^2\mathrm{Im}\tau$ piece in (4.5c). What remains is $1/4\pi\mathrm{Im}\tau$.

- The integral over $\mathrm{Im}\tau$ is done from 0 to ∞. We will have to regulate the UV divergences coming from the region of integration around $t = 0$. We will use for simplicity the Schwinger regularization which amounts to integrating the parameter t in the interval $[1/\pi\Lambda^2, \infty]$.

Then,

$$\frac{16\pi^2}{g_i^2}|_{1-loop}^{EFT} = L_1 + L_2 + L_3 \tag{5.16}$$

where

$$L_1 = -\frac{1}{\pi} Str\left[\bar{\mathcal{P}}_i^2 \left(\frac{1}{12} - \chi^2\right)\right] \mathbf{K}_1(\mu_e) + \frac{k_i}{8\pi^2} Str\left[\frac{1}{12} - \chi^2\right] \mathbf{K}_2(\mu_e) \tag{5.17}$$

$$L_2 = -\frac{1}{4\pi^2}\left(1 + \frac{1}{3\mu_e^2}\right) Str[\bar{\mathcal{P}}_i^2]\mathbf{K}_2(\mu_e) + \frac{k_i}{16\pi^3}\left(\frac{1}{2} + \frac{1}{3\mu_e^2}\right) Str[\mathbf{1}]\mathbf{K}_3(\mu_e) \tag{5.18}$$

$$L_3 = -\frac{k_i(1 + 2\mu_e^2)}{64\pi^3\mu_e^2} Str[\mathbf{1}]\mathbf{K}_3(\mu_e) \tag{5.19}$$

and

$$\mathbf{K}_n(\mu_e) \equiv \frac{1}{V(\mu_e)} \int_{\frac{1}{\pi\Lambda^2}}^{\infty} \frac{dt}{t^n} \partial_{\mu_e^2} \sqrt{t} \left[\theta_3(it\mu_e^2) - \theta_3(4it\mu_e^2)\right] \tag{5.20}$$

The integrals can again be evaluated

$$\mathbf{K}_1(\mu_e, \Lambda) = 4\pi \log(\mu_e/\Lambda) + 2\pi(\gamma_E - 2) + \mathcal{O}\left(e^{-\Lambda^2/\mu_e^2}\right) \tag{5.21}$$

and for $n > 1$

$$\mathbf{K}_n(\mu_e, \Lambda) = \frac{2\pi\Lambda^{2n-2}}{1-n} + 8\pi^{2-n}(2n-3)(1 - 2^{2n-3})\Gamma(n-1)\zeta(2n-2)\mu_e^{2n-2} + \mathcal{O}\left(e^{-\Lambda^2/\mu_e^2}\right) \tag{5.22}$$

In a similar fashion we can calculate the string one-loop correction to the R^2 coupling with the result

$$\frac{1}{g_{R^2}^2}|_{1-loop} = \frac{4}{\pi V(\mu)} \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\tau |\eta|^4} \sum_{a,b}^{1} \left[\partial_\tau \left(\frac{\theta[^a_b]}{\eta}\right) \left(\bar{G}_2 - \frac{1}{6\mu^2}\partial_{\bar{\tau}}\right) X' \tag{5.23}\right.$$

$$+\frac{1}{6\mu^2}\frac{\theta[^a_b]}{\eta}\left(\bar{G}_2-\frac{1}{6\mu^2}\partial_{\bar\tau}\right)\partial_\tau X'\Bigg]+\frac{k(k+2)}{16\pi V(\mu)}\int_{\mathcal{F}}\frac{d^2\tau}{\mathrm{Im}\tau^2}X'Z_0$$

where

$$\bar{G}_2\equiv\partial_{\bar\tau}\log\bar\eta+\frac{i}{4\mathrm{Im}\tau}=\frac{1}{2}\partial_{\bar\tau}\log[\mathrm{Im}\tau\bar\eta^2] \tag{5.24}$$

One-loop corrections to higher dimension operators can also be computed. We give here the result for $F^4_{\mu\nu}$ for $Z_2\times Z_2$ symmetric orbifold compactifications of the heterotic string. This correction gets contributions from all sectors including $N=4$ ones and it is thus interesting for studying decompactification problems in string theory. The $N=4$ sector contribution to the $F^4_{\mu\nu}$ term for the E_8 gauge group can be computed from (4.9) to be

$$\frac{1}{g^2_{F^4}}|^{E_8}_{1-loop}=\frac{1}{V(\mu)}\int_{\mathcal{F}}\frac{d^2\tau}{\mathrm{Im}\tau^2}X'(\mu)\prod_{i=1}^{3}[\mathrm{Im}\tau\Gamma_{2,2}(T_i,U_i)]\sum_{a,b=0}^{1}\frac{\bar\vartheta^8[^a_b]}{\bar\eta^{24}}\times \tag{5.25}$$

$$\times\sum_{\gamma,\delta=0}^{1}\bar\vartheta^7[^\gamma_\delta]\left(\frac{i}{\pi}\partial_{\bar\tau}-\frac{5}{2\pi\mathrm{Im}\tau}\right)\left(\frac{i}{\pi}\partial_{\bar\tau}-\frac{1}{4\pi\mathrm{Im}\tau}\right)\bar\vartheta[^\gamma_\delta]$$

6 IR Flow Equations for Couplings

Once we have obtained the one-loop corrections to the coupling constants, we can observe that they satisfy scaling type flows. We will present here IR Flow Equations (IRFE) for differences of gauge couplings.

The existence of IRFE is due to differential equations satisfied by the lattice sum of an arbitrary (d,d) lattice,

$$Z_{d,d}=\mathrm{Im}\tau^{d/2}\sum_{P_L,P_R}e^{i\pi\tau P_L^2/2-i\pi\bar\tau P_R^2/2} \tag{6.1}$$

where

$$P^2_{L,R}=\vec{n}G^{-1}\vec{n}+2\vec{m}BG^{-1}\vec{n}+\vec{m}[G-BG^{-1}B]\vec{m}\pm 2\vec{m}\cdot\vec{n} \tag{6.2}$$

$\vec{m},\vec{n}$ are integer d-dimensional vectors and G_{ij} (B_{ij}) is a real symmetric (antisymmetric) matrix. $Z_{d,d}$ is $O(d,d,Z)$ and modular invariant. Moreover it satisfies the following second order differential equation*:

$$\left[\left(G_{ij}\frac{\partial}{\partial G_{ij}}+\frac{1-d}{2}\right)^2+2G_{ik}G_{jl}\frac{\partial^2}{\partial B_{ij}\partial B_{kl}}-\frac{1}{4}-4\mathrm{Im}\tau^2\frac{\partial^2}{\partial\tau\partial\bar\tau}\right]Z_{d,d}=0 \tag{6.3}$$

The equation above involves also the modulus of the torus τ. Thus it can be used to convert the integrands for threshold corrections to differences of coupling constants into total derivatives on τ-moduli space. We will focus on gauge couplings of $Z_2\times Z_2$ orbifold models. To derive such an equation we start from the integral expressions of such couplings (5.3) to obtain

$$\Delta_{AB}\equiv\frac{16\pi^2}{g_A^2}-\frac{16\pi^2}{g_B^2}=-4\mu^3(b_A-b_B)\int_{\mathcal{F}}\frac{d^2\tau}{\mathrm{Im}\tau^2}X'(\mu)\mathrm{Im}\tau\Gamma_{2,2}(T,\bar{T},U,\bar{U}) \tag{6.4}$$

*The special case for $d=2$ of this equation was noted and used in [9, 11].

Eq. (6.4) does not apply to $U(1)$'s that can get enhanced at special points of the moduli. Using (6.3) we obtain

$$\left[\left(\mu\frac{\partial}{\partial\mu}\right)^2 - 2\mu\frac{\partial}{\partial\mu} - 16\mathrm{Im}T^2\frac{\partial^2}{\partial T\partial\bar{T}}\right]\Delta_{AB} = 0 \tag{6.5}$$

and we have also a similar one with $T \to U$. Note that for couplings that have a logarithmic behavior, the double derivative of μ does not contribute.

We strongly believe that such equations also exist for single coupling constants using appropriate differential equations for $(d, d+n)$ lattices.

Notice that the IR scale μ plays the role of the RG scale in the effective field theory:

$$\frac{16\pi^2}{g_A^2(\mu)} = \frac{16\pi^2}{g_A^2(M_{str})} + b_A\log\frac{M_{str}^2}{\mu^2} + F_A(T_i) + \mathcal{O}(\mu^2/M_{str}^2) \tag{6.6}$$

where the moduli T_i have been rescaled by M_{str} so they are dimensionless. Second, the IRFE gives a scaling relation for the moduli dependent corrections. Such relations are very useful for determining the moduli dependence of the threshold corrections. We will illustrate below such a determination, applicable to the $Z_2 \times Z_2$ example described above.

Using the expansion (6.6) and applying the IRFE (6.5) we obtain

$$\mathrm{Im}T^2\frac{\partial^2}{\partial T\partial\bar{T}}(F_A - F_B) = \frac{1}{4}(b_A - b_B) \tag{6.7}$$

and a similar one for U. This non-homogeneous equation has been obtained in [9, 11].

Solving them we obtain

$$F_A - F_B = (b_B - b_A)\log[\mathrm{Im}T\mathrm{Im}U] + f(T,U) + g(T,\bar{U}) + \mathrm{cc} \tag{6.8}$$

If at special points in moduli space, the extra massless states are uncharged with respect to the gauge groups appearing in (6.7) then the functions f and g are non-singular inside moduli space. In such a case duality invariance of the threshold corrections implies that

$$F_A - F_B = (b_B - b_A)\log[\mathrm{Im}T\mathrm{Im}U|\eta(T)\eta(U)|^4] + \mathrm{constant} \tag{6.9}$$

This is the result obtained via direct calculation in [9].

It is thus obvious that the IRFE provides a powerful tool in evaluating general threshold corrections as manifestly duality invariant functions of the moduli.

7 Further Directions

Another set of important couplings that we have not explicitly addressed in this paper are the Yukawa couplings. Physical Yukawa couplings depend on the Kähler potential and the superpotential. The superpotential receives no perturbative contributions and thus can be calculated at tree level. The Kähler potential however does get renormalized so in order to compute the one-loop corrected Yukawa couplings we have to compute the one-loop renormalization of the Kähler metric. When the ground state has (spontaneously-broken) spacetime supersymmetry the wavefunction renormalization of the scalars ϕ_i are the same as those for their auxiliary fields F_i. Thus we need to turn on non-trivial F_i, calculate their effective action on the torus and pick the quadratic part proportional to $F_i\bar{F}_{\bar{j}}$. This can be easily done using the techniques we developed in this paper since it turns out that the vertex operators [26] for some relevant F fields are bilinears of left and right U(1) chiral currents.

There are several other open problems that need to be addressed in this context.

The structure of higher loop corrections should be investigated. A priori there is a potential problem, due to the dilaton, at higher loops. One would expect that since there is a region of spacetime where the string coupling become arbitrarily strong, higher order computations would be problematic. We think that this is not a problem in our models, because in Liouville models with N=4 superconformal symmetry (which is the case we consider) there should be no divergence due to the dilaton at higher loops. However, this point need further study. One should eventually analyze the validity of non-renormalization theorems at higher loops [11] since they are of prime importance for phenomenology.

The consequences of string threshold corrections for low energy physics should be studied in order to be able to make quantitative predictions.

Finally, with respect to magnetic and gravitational instabilities, more study is needed in order to draw model-independent conclusions on the perturbative stability of string ground states. This is important since it provides the only perturbative way to investigate stability in a first quantized formulation.

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