CERN-TH/95-183
hepth@xxx/9507011

BPS States, Weight Spaces and Vanishing Cycles

W. Lerche

CERN, CH 1211 Geneva 23, Switzerland

Contribution to the Proceedings of
Strings '95, USC, Los Angeles,
and to the Proceedings of the Trieste Conference on
S-Duality and Mirror Symmetry, 1995

We review some simple group theoretical properties of BPS states, in relation with the singular homology of level surfaces. Primary focus is on classical and quantum $N=2$ supersymmetric Yang-Mills theory, though the considerations can be applied to string theory as well.

CERN-TH/95-183
July 1995

BPS STATES, WEIGHT SPACES AND VANISHING CYCLES

W. LERCHE
CERN, Geneva, Switzerland

ABSTRACT

We review some simple group theoretical properties of BPS states, in relation with the singular homology of level surfaces. Primary focus is on classical and quantum $N=2$ supersymmetric Yang-Mills theory, though the considerations can be applied to string theory as well.

There has been dramatic recent progress in understanding non-perturbative properties of certain supersymmetric field[1–9] and string[10–16] theories. A common feature of such theories are singularities in the quantum moduli space, arising from certain BPS states becoming massless in these vacuum parameter regions. Much of the present discussion is centered at the properties of such states. Such states typically have various different representations, for example, a representation in terms of ordinary elementary fields, or, in a dual formulation, in terms of non-perturbative solitonic bound states[17].

We will first discuss some generic properties of BPS states, with main emphasis on $N=2$ supersymmetric theories. We will then concretely specialize further below to classical and quantum Yang-Mills theory.

The mass of a BPS state is directly given[18] in terms of the central charge Z of the relevant underlying $N=2$ or $N=4$ supersymmetry algebra,

$$m^2 \simeq |Z|^2 , \qquad Z \equiv \vec{q}\cdot\vec{a} + \vec{g}\cdot\vec{a}_D . \tag{1}$$

Here, $\vec{q},\vec{g}$ are the electric and magnetic charges of the state in question, and $\vec{a},\vec{a}_D$ are the classical values of the Higgs field and "magnetic dual" Higgs field, respectively. An important insight[1] is that Z can be represented as a period integral of some prime form λ on a suitable "level" surface X, ie.,

$$Z = \int_\nu \lambda , \tag{2}$$

where ν is a cycle in the middle homology of X, $\nu \in H_{dim_C(X)}(X,\mathbb{Z})$. Obviously, $Z=0$ if $\nu=0$ (since λ does not blow up), so that the masslessness of a BPS state can be attributed to a collapsed "vanishing cycle" on X. Such vanishing cycles shrink to zero in certain regions $\Sigma(X)$ of the moduli space $\mathcal{M}(X)$, and it was the idea of Seiberg and Witten[1] to literally identify $\mathcal{M}(X)$ with the quantum moduli space of the physical model in question. Typically, the singular locus $\Sigma(X)$ consists of branches of codimension one, and on any such branch just one cycle degenerates, see Fig.1; of course, on intersections of such branches the surface X becomes singular in a more non-trivial way.

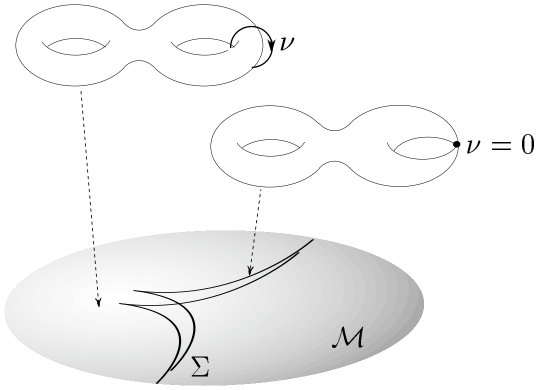

Fig.1: On the singular locus $\Sigma(X)$ in moduli space $\mathcal{M}$, the level surface X degenerates by pinching of vanishing cycles ν. The coordinates of any such cycle with respect to some symplectic basis of $H_{dim_C(X)}(X, \mathbb{Z})$ gives the electric and magnetic quantum numbers of the corresponding massless BPS state. Shown here is the genus two Riemann surface associated with $SU(3)$ $N=2$ supersymmetric Yang-Mills theory.

It turns out that many features of the massless BPS spectrum can directly be studied in terms of the singular homology of X. Note, however, that various properties crucially depend on whether $d \equiv dim_C(X)$ is even or odd.

Odd d is the situation of quantum $N=2$ supersymmetric Yang-Mills theory, where X is a hyperelliptic curve[2,3], and of $N=2$ supersymmetric type II string compactifications, where X is a Calabi-Yau manifold[12,13,15]. In these theories, the massless BPS states are given by "matter" hypermultiplets. For this case of odd d, the intersection form Ω of H_d is skew-symmetric, so that there is a natural split of H_d into two sets of non-intersecting cycles, whose bases may be denoted by γ_α and γ_β. This split just corresponds to distinguishing electric and magnetic degrees of freedom. Specifically, a vanishing cycle can be expanded as follows:

$$\nu \;=\; \vec{q}\cdot\vec{\gamma}_\alpha + \vec{g}\cdot\vec{\gamma}_\beta\,, \qquad\qquad \gamma_\alpha, \gamma_\beta \in H_{dim_C(X)}(X, \mathbb{Z})\ . \tag{3}$$

Identifying $\vec{a} = \int_{\vec{\gamma}_\alpha}\lambda$, $\vec{a}_D = \int_{\vec{\gamma}_\beta}\lambda$, we immediately see that the electric and magnetic quantum numbers $\vec{q}, \vec{g}$ of a massless BPS state are simply given by the coordinates of the corresponding vanishing cycle[7]. Obviously, under a change of homology basis, the charges change as well, but this is nothing but a duality rotation. What remains invariant is the intersection number

$$\nu_i \circ \nu_j \;=\; \vec{\nu}^t\cdot\Omega\cdot\vec{\nu} \;= \langle \vec{g}_i, \vec{q}_j\rangle - \langle \vec{g}_j, \vec{q}_i\rangle \;\in\; \mathbb{Z}\ , \tag{4}$$

where Ω is a symplectic metric, and $\langle\ ,\ \rangle$ is the inner product in weight space. (If we take both electric and magnetic charge vectors in simple root bases, then $\Omega = \left(\begin{smallmatrix} 0 & C \\ -C & 0\end{smallmatrix}\right)$, where C is a Cartan matrix, or some generalization of it).

Note that (4) represents the well-known Dirac-Zwanziger quantization condition for the possible electric and magnetic charges, and we see that it satisfied by construction. The vanishing of the r.h.s. of (4) is required for two states to be local with respect to each other[19,5]. Thus, only states that are related to non-intersecting cycles are mutually local, and can be simultaneously represented in a local effective lagrangian. For example, 't Hooft-Polyakov monopoles (typically associated with γ_β-cycles) are not local with respect to elementary gauge bosons (associated with the dual, intersecting γ_α-cycles).

Quintessential for the solution of $N=2$ Yang-Mills theory[1] was the study of the global non-abelian monodromy properties of the quantum moduli space. Specifically, we noted that there is a particular vanishing cycle ν associated with each branch of $\Sigma(X)$. The monodromy action on any given cycle, $\gamma \in H_d(X, \mathbb{Z})$, is directly determined in terms of this vanishing cycle by means of the Picard-Lefshetz formula[20]:

$$M_\nu : \quad \delta \longrightarrow \delta - (\delta \circ \nu)\, \nu \ , \tag{5}$$

where $\circ$ is the intersection product. From this one can find for a cycle of the form (3) the following monodromy matrix[7]:

$$M_{\nu=(g,q)} = \begin{pmatrix} \mathbb{1} + \vec{q} \otimes \vec{g} & \vec{q} \otimes \vec{q} \\ -\vec{g} \otimes \vec{g} & \mathbb{1} - \vec{g} \otimes \vec{q} \end{pmatrix} \ . \tag{6}$$

It directly expresses the correct logarithmic shift property of the $N=2$ quantum effective action $\mathcal{F}(a)$, and thus automatically guarantees a consistent physical picture. That is, near the vanishing of some ν, the monodromy shift of the gauge coupling $\tau \equiv \partial_a^2 \mathcal{F}(a)$, when expressed in suitable local variables, is

$$\Delta_\nu \tau_{ij} \ = \frac{\partial}{\partial a_i} \Delta_\nu a_{D,j} \ = \ -(\gamma_j^* \circ \nu) \frac{\partial}{\partial a_i} Z_\nu \ = \ -\nu_i\, \nu_j \ , \tag{7}$$

where γ^* is the cycle dual to γ. This is indeed precisely the monodromy associated with the corresponding one-loop $N=2$ effective action in the local patch near the singular line $\Sigma(X)^{(\nu)}$:

$$\mathcal{F}_\nu \ = \ \frac{1}{4\pi i} {Z_\nu}^2 \log\big[\frac{Z_\nu}{\Lambda}\big] \ .$$

On the other hand, if d is even, the intersection metric Ω is symmetric, and there is no natural distinction between "electric" and "magnetic" cycles (typically, Ω is given by a Cartan matrix). This is essentially the classical, self-dual situation, where there are non-abelian gauge bosons among the massless BPS states. Specifically, $d=0$ corresponds to classical $N=2$ Yang-Mills theory, and $d=2$ corresponds to type II supersymmetric string compactifications[10,11,14], with $X = K_3$.

Most of the above formulas apply just as well (up to a sign change in the PL formula), if we simply drop the magnetic variables a_D, γ_b. Since now the monodromy matrices do not link electric and magnetic sectors, the PL monodromy does not describe block off-diagonal logarithmic shifts that were attributed before to perturbative corrections, but rather gives directly discrete gauge (Weyl group) transformations – this indeed reflects properties of a theory without quantum corrections.

To summarize the general scheme, we present the following suggestive table, where C denotes the Cartan matrix, $\Lambda_{R,W}$ root and weight lattices, W a simple singularity, and a canonical basis of the vanishing cycles has been chosen. One clearly recognizes the alternating pattern depending on whether d is even or odd.

phys. model	X	$d(X)$	$H_d(X,\mathbb{Z})$	int. form	BPS states
class. SYM	$W=0$	0	Λ_R	C	gauge bosons
quant. SYM	$W^2-1=y^2$	1	$\Lambda_W\otimes\Lambda_R$	$\left(\begin{smallmatrix}0 & C\\ -C & 0\end{smallmatrix}\right)$	dyons
type II comp	K_3	2	$\Lambda_R(E_8)^2\otimes\Lambda_{1,1}{}^3$	$C_{E_8}{}^2\otimes C_{1,1}{}^3$	gauge solitons
type II comp	Calabi–Yau	3	...	...	dyonic bl holes

Since classical $N=2$ Yang-Mills theory is the simplest example, we like now to discuss it in somewhat more detail[7], focusing mainly on gauge group $G=SU(n)$.

The scalar superfield component ϕ labels a continuous family of inequivalent ground states, which constitutes the classical moduli space, $\mathcal{M}_0$. One can always rotate it into the Cartan sub-algebra, $\phi=\sum_{k=1}^{n-1}a_kH_k$, with $H_k=E_{k,k}-E_{k+1,k+1}$, $(E_{k,l})_{i,j}=\delta_{ik}\delta_{jl}$. For generic eigenvalues of ϕ, the $SU(n)$ gauge symmetry is broken to the maximal torus $U(1)^{n-1}$, whereas if some eigenvalues coincide, some larger, non-abelian group $H\subseteq G$ remains unbroken. Precisely which gauge bosons are massless for a given background $\vec{a}=\{a_k\}$, can easily be read off from the central charge formula: $Z_q(a)\equiv\vec{q}\cdot\vec{a}$, where we take for the charge vectors $\vec{q}$ the roots $\vec{\alpha}\in\Lambda_R(G)$ in Dynkin basis.

The Cartan sub-algebra variables a_k are not gauge invariant and in particular not invariant under discrete Weyl transformations. Therefore, one introduces other variables for parametrizing the classical moduli space, which are given by the Weyl invariant Casimirs $u_k(a)$. These variables parametrize the Cartan sub-algebra modulo the Weyl group, ie, $\{u_k\}\cong\mathbb{C}^{n-1}/S(n)$, and can be obtained by a Miura transformation:

$$\prod_{i=1}^{n}\big(x-Z_{\lambda_i}(a)\big)\;=\;x^n-\sum_{l=0}^{n-2}u_{l+2}(a)\,x^{n-2-l}\;\equiv\;W_{A_{n-1}}(x,u)\,. \qquad (8)$$

Here, λ_i are the weights of the n-dimensional fundamental representation, and $W_{A_{n-1}}(x,u)$ is nothing but the simple singularity[20] associated with $SU(n)$, with

$$u_k(a)\;=\;(-1)^{k+1}\sum_{j_1\neq\ldots\neq j_k}Z_{\lambda_{j_1}}Z_{\lambda_{j_2}}\ldots Z_{\lambda_{j_k}}(a)\,. \qquad (9)$$

These symmetric polynomials are manifestly invariant under the Weyl group $S(n)$, which acts by permutation of the weights λ_i.

From the above we know that whenever $Z_{\lambda_i}(a) = Z_{\lambda_j}(a)$ for some i and j, there are, classically, extra massless non-abelian gauge bosons, since $Z_\alpha = 0$ for some root α. For such backgrounds the effective action becomes singular. The classical moduli space is thus given by the space of Weyl invariant deformations modulo such singular regions: $\mathcal{M}_0 = \{u_k\}\backslash\Sigma_0$. Here, $\Sigma_0 \equiv \{u_k : \Delta_0(u_k) = 0\}$ is the zero locus of the "classical" discriminant

$$\Delta_0(u) \;=\; \prod_{i<j}^{n}(Z_{\lambda_i}(u) - Z_{\lambda_j}(u))^2 \;=\; \prod_{\substack{\text{positive}\\ \text{roots }\alpha}} (Z_\alpha)^2(u) \;, \tag{10}$$

of the simple singularity (8). We schematically depicted the singular loci Σ_0 for $n = 2, 3, 4$ in Fig.2.

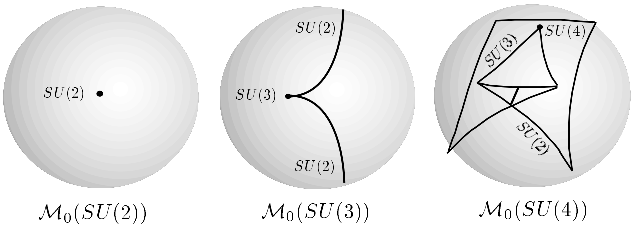

Fig.2: Singular loci Σ_0 in the classical moduli spaces $\mathcal{M}_0$ of pure $SU(n)$ $N=2$ Yang-Mills theory. They are nothing but the bifurcation sets of the type A_{n-1} simple singularities, and reflect all possible symmetry breaking patterns in a gauge invariant way (for $SU(3)$ and $SU(4)$ we show only the real parts). The picture for $SU(4)$ is known in singularity theory as the "swallowtail".

The discriminant loci Σ_0 are generally given by intersecting hypersurfaces of complex codimension one. On each such surface one has $Z_\alpha = 0$ for some pair of roots $\pm\alpha$, so that there is an unbroken $SU(2)$. Furthermore, since $Z_\alpha = 0$ is a fixed point of the Weyl transformation r_α, the Weyl group action is singular on these surfaces. On the intersections of these surfaces one has, correspondingly, larger unbroken gauge groups. All planes together intersect in just one point, namely in the origin, where the gauge group $SU(n)$ is fully restored. Thus, what we learn is that all possible classical symmetry breaking patterns are encoded in the discriminants of $W_{A_{n-1}}(x, u)$.

For classical $SU(n)$ $N = 2$ Yang-Mills theory, the relevant level surface X is zero dimensional and given by the following set of points:

$$X \;=\; \{\, x : W_{A_{n-1}}(x, u) = 0 \,\} \;=\; \{\, Z_{\lambda_i}(u) \,\} \,. \tag{11}$$

It is singular if any two of the $Z_{\lambda_i}(u)$ coincide, and indeed, the vanishing cycles are just given by the differences: $\nu_\alpha = Z_{\lambda_i} - Z_{\lambda_j} = Z_\alpha$, ie., by the central charges associated with the non-abelian gauge bosons. It is indeed well-known[20] that ν_α generate the root lattice: $H_0(X, \mathbb{Z}) \cong \Lambda_R$. We depicted the level surface for $G = SU(3)$ in Fig.3.

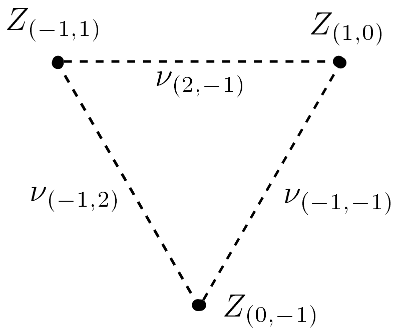

Fig.3: Level manifold for classical $SU(3)$ Yang-Mills theory, given by points in the x-plane; they form the projection of a weight diagram. The dashed lines are the vanishing cycles associated with non-abelian gauge bosons (having corresponding quantum numbers, here in Dynkin basis). The masses are proportional to the lengths of the lines and thus vanish if the cycles collapse.

Pictures like this one have a very concrete group theoretical meaning. In fact, if we choose (as in Fig.3) a special region in the moduli space where only the top Casimir, u_n, is non-zero, the picture becomes precisely the projection of the weights λ_i (that live in the $n-1$-dimensional weight space) to the two-dimensional eigenspace of the Coxeter element with $\mathbb{Z}_n$ action (why this so is follows from the considerations of ref.[21], where group theoretical aspects of $u_k = 0, k \neq n$ are discussed).

Thus, we see another manifestation of the close connection between the vanishing homology of X and $SU(n)$ weight space. The intersection numbers of the vanishing cycles are just given by the inner products between root vectors, $\nu_{\alpha_i} \circ \nu_{\alpha_j} = \langle \alpha_i, \alpha_j \rangle$ (self-intersections counting $+2$), and the Picard-Lefshetz formula (5) coincides in this case with the well-known formula for Weyl reflections, with matrix representation: $M_{\alpha_i} = \mathbb{1} - \alpha_i \otimes w_i$ (where w_i are the fundamental weights).

Note that the corresponding situation in classical string theory[10,11] is when one takes $X = K_3$ and replaces the euclidean $SU(n)$ weight space with the lorentzian Narain lattice, which is isomorphic to the lattice of 2-cycles of the K_3 surface.

Most of the above considerations apply more or less directly to the other simply laced Lie groups of type D and E, for which the following simple singularities [20] are relevant:[⋆]

⋆ For non-simply laced gauge groups B_n, the corresponding boundary singularities are relevant; cf., see the discussion of the quantum theory in ref.[6].

$$
\begin{aligned}
W_{D_n}(x_1,x_2,u) &= {x_1}^{n-1} + \tfrac{1}{2}x_1\,{x_2}^2 - \sum_{l=1}^{n-1} u_{2l}\,{x_1}^{n-l-1} - \widetilde{u}_n x_2 \\
W_{E_6}(x_1,x_2,u) &= {x_1}^3 + {x_2}^4 - u_2x_1x_2^2 - u_5x_1x_2 - u_6x_2^2 - u_8x_1 - u_9x_2 - u_{12} \\
W_{E_7}(x_1,x_2,u) &= {x_1}^3 + x_1{x_2}^3 - u_2x_1^2x_2 - u_6x_1^2 - u_8x_1x_2 - u_{10}x_2^2 \\
&\quad - u_{12}x_1 - u_{14}x_2 - u_{18} \\
W_{E_8}(x_1,x_2,u) &= {x_1}^3 + {x_2}^5 - u_2x_1x_2^3 - u_8x_1x_2^2 - u_{12}x_2^3 - u_{14}x_1x_2 \\
&\quad - u_{18}x_2^2 - u_{24}x_2 - u_{30}\ ,
\end{aligned}
\tag{12}
$$

The discriminants of these singularities give indeed precisely the singularities in the corresponding classical Yang-Mills moduli spaces.

Note, though, that the number of variables of these singularities is two, and not one as for A_{n-1}. Certain features, like the intersection properties of vanishing cycles, depend critically on the number of variables. One usually stabilizes the situation by adding irrelevant quadratic pieces to the singularities, so that all ADE singularities are represented by three variables. This would however confuse the situation in the present context, and one rather prefers to have only one variable, x.

For D_n this can easily be achieved[9] by integrating out x_2 using its "equations of motion", $\partial_{x_2}W = 0$ (just as it is known from the LG description of type D_n $N=2$ superconformal minimal models[22]). The resulting level surface,

$$
X:\quad \widetilde{W}_{D_n}(x) \equiv x^2 W_{D_n}(x_1 = x^2, x_2 = \widetilde{u}_n/x^2)\ =\ 0\ ,
$$

gives (for $u_l = 0, l \neq 2n-2$) indeed the projection of the weight diagram of the vector representation, see Fig.4 for an example. Note that the correct interpretation of these surfaces is more involved. Specifically, even though the origin $x = 0$ has a double zero, no massless gauge bosons are associated with it, because this "zero distance" does not correspond to a projection of a root vector (the two weights that project to the origin differ by sums of roots, and this rather corresponds to a multi-particle state). The correct surface is obtained by modding out by $x \to -x$, as each root appears twice in the picture. Taking this into account by appropriately considering only $\mathbb{Z}_2$-odd linear combinations of cycles, one obtains the correct singular behavior.[9]

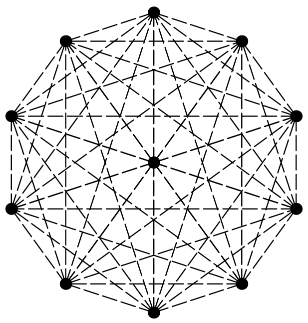

Fig.4: Level surface for classical D_6 Yang-Mills theory, in analogy to Fig.3. It takes this Coxeter-symmetric form if only the top Casimir is non-zero. The lines represent the vanishing cycles and are projections of root vectors. Note that each of the 30 positive roots occurs twice, and that the center is two-fold degenerate. This kind of pictures has appeared before in the discussion of integrable two-dimensional $N=2$ LG models[21], where, precisely as here, the line lengths give the masses of BPS states (they are given by the components of the Frobenius-Perron eigenvector of the Cartan matrix).

How this is precisely to be done for the exceptional groups is less clear at the moment, but in analogy one expects that after eliminating x_2 from the exceptional simple singularities, one obtains expressions $\widetilde{W}_{E_n}(x,u)$ whose orders are given by the dimensions of the defining representation, and which might have a close relationship to Lax operators obtained from Drinfel'd-Sokolov reduction. If true, the level surfaces for E_n would look like the pictures given in ref.[21].

We now turn to the quantum version of the $N=2$ Yang-Mills theories, where the issue is to construct curves X whose moduli spaces $\mathcal{M}_\Lambda$ give the supposed quantum moduli spaces. We have seen that the classical theories are characterized by simple singularities, so we may expect that the quantum versions should also have something to do with them. Indeed, for $G=SU(n)$ the appropriate manifolds were found in[2,3] and are given by

$$X: \quad y^2 \;=\; \big(W_{A_{n-1}}(x,u_i)\big)^2 - \Lambda^{2n} \quad , \tag{13}$$

which corresponds to special genus $g=n-1$ hyperelliptic curves. Above, Λ is the dynamically generated quantum scale.

Since y^2 factors into $W_{A_{n-1}} \pm \Lambda^n$, the situation is in some respect like two copies of the classical theory, with the top Casimir u_n shifted by $\pm\Lambda^n$. Specifically, the "quantum" discriminant, whose zero locus Σ_Λ gives the singularities in the quantum moduli space $\mathcal{M}_\Lambda$, is easily seen to factorize as follows:

$$\begin{aligned}\Delta_\Lambda(u_k,\Lambda) &\equiv \prod_{i<j}(Z^+_{\lambda_i}-Z^+_{\lambda_j})^2(Z^-_{\lambda_i}-Z^-_{\lambda_j})^2 = \text{const.}\,\Lambda^{2n^2}\delta_+\,\delta_-\ , \quad \text{where} \\ \delta_\pm(u_k,\Lambda) &= \Delta_0(u_2,...,u_{n-1},u_n\pm\Lambda^n)\ ,\end{aligned} \tag{14}$$

is the shifted classical discriminant (10). Thus, Σ_Λ consists of two copies of the classical singular locus Σ_0, shifted by $\pm\Lambda^n$ in the u_n direction. Obviously, for $\Lambda\to 0$, the classical moduli space is recovered: $\Sigma_\Lambda\to\Sigma_0$. That is, when the quantum corrections are switched on, a single isolated branch of Σ_0 (associated with massless gauge bosons of a particular $SU(2)$ subgroup) splits into two branches of Σ_Λ (reflecting two massless Seiberg-Witten dyons related to this $SU(2)$). This is depicted in Fig.5.

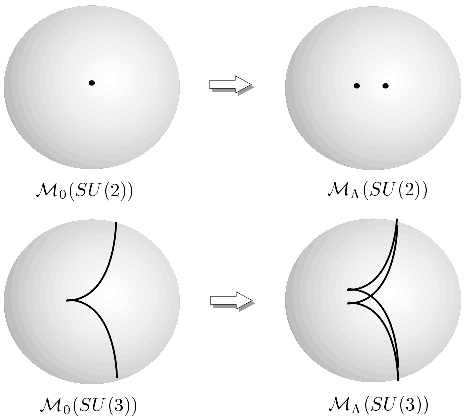

Fig.5: When switching to the exact quantum theory, the classical singular locus splits into two quantum singularities that are associated with massless dyons. The distance is governed by the quantum scale Λ.

Moreover, the points Z_{λ_i} of the classical level surface (11) split as follows,

$$Z_{\lambda_i}(u)\ \to\ Z^\pm_{\lambda_i}(u,\Lambda)\equiv\ Z_{\lambda_i}(u_2,,...,u_{n-1},u_n\pm\Lambda^n)\ , \tag{15}$$

and become the $2n$ branch points of the Riemann surface (13). The curve can thus be represented by the two-sheeted x-plane with cuts running between pairs $Z^+_{\lambda_i}$ and $Z^-_{\lambda_i}$. See Fig.6 for an example.

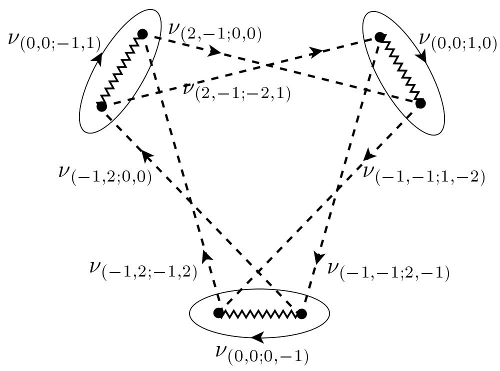

Fig.6: The level manifold of quantum $SU(3)$ Yang-Mills theory is given by a genus two Riemann surface, which is represented here as a two-sheeted cover of the x-plane. It may be thought as the quantum version of the classical, zero dimensional level surface of Fig.3, whose points transmute into pairs of branch points. The dashed lines represent the vanishing cycles (on the upper sheet) that are associated with the six branches of the singular locus Σ_Λ. The quantum numbers refer to $(\vec{g};\vec{q})$, where $\vec{g},\vec{q}$ are weight vectors in Dynkin basis.

Just like for the classical level surfaces, the vanishing cycles of the Riemann surfaces (13) have a concrete group theoretical meaning. Not only may one expect to determine the quantum numbers of the massless dyons by just expanding the vanishing cycles in some appropriate symplectic basis, one find that one can also directly associate the cycles in the branched x-plane with projections of roots and weights.

Specifically, Fig.6 can be thought of as a quantum deformation of the classical level surface in Fig.3, whose points, associated with projected weight vectors λ_i, turn into branch cuts (whose length is governed by the quantum scale, Λ). In fact, one obtains two, slightly rotated copies of the weight diagram. A basis of cycles can be chosen such that the coordinates of the "electric", γ_α-type of cycles are given by precisely the weight vectors λ_i. Moreover, the classical cycles of Fig.3 turn into pairs of "magnetic", γ_β-type of cycles, and we can immediately read off the electric and magnetic quantum numbers of the massless dyons (note that they are given by root vectors). Accordingly, the intersection properties of the cycles are reflected by symplectic inner products in the two copies of weight space.

These considerations apply to D_n gauge groups as well; here, the relevant quantum surfaces[9] are given by analogous deformations of the classical level surfaces (cf., Fig.4), and one needs to project on Z_2-odd cycles.

Acknowledgements

I would like to thank the organizers of the conference for their hard work, and A. Klemm, S. Theisen and S. Yankielowicz for enjoyable collaboration. I also thank A. Brandhuber and K. Landsteiner for sharing their insights.

References

1. N. Seiberg and E. Witten, *Nucl. Phys.* **B426** (1994) 19, hep-th/9407087; *Nucl. Phys.* **B431** (1994) 484, hep-th/9408099.
2. A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, *Phys. Lett.* **B344** (1995) 169, hep-th/9411048.
3. P. Argyres and A. Faraggi, *Phys. Rev. Lett.* **73** (1995) 3931, hep-th/9411057.
4. M. Douglas and S. Shenker, *Dynamics of SU(N) Supersymmetric Gauge Theory*, preprint RU-95-12, hep-th/9503163.
5. P. Argyres and M. Douglas, *New Phenomena in SU(3) Supersymmetric Gauge Theory*, preprint IASSNS-HEP-95/31, RU-95-28, hep-th/9505062.
6. U. Danielsson and B. Sundborg, *The Moduli Space and Monodromies of N=2 Supersymmetric SO(2r+1) Yang-Mills Theory*, preprint USITP-95-06, UUITP-4/95, hep-th/9504102.
7. A. Klemm, W. Lerche and S. Theisen, *Nonperturbative Effective Actions of N=2 Supersymmetric Gauge Theories*, preprint CERN-TH/95/104, LMU-TPW 95-7, hep-th/9505150.
8. A. Hanany and Y. Oz, *On the Quantum Moduli Space of N=2 Supersymmetric SU(Nc) Gauge Theories*, preprint TAUP-2248-95,WIS-95/19/May-PH, hep-th/9505075; P. Argyres, M. Plesser and A. Shapere, *The Coulomb Phase of N=2 Supersymmetric QCD*, preprint IASSNS-HEP-95/32, UK-HEP/95-06, hep-th/9505100.
9. A. Brandhuber and K. Landsteiner, *On the Monodromies of N=1 Supersymmetric Yang-Mills Theory with Gauge Group SO(2n)*, preprint CERN-TH/95-180, hep-th/9507008.
10. C. Hull and P. Townsend, *Unity of Superstring Dualities*, preprint QMW-94-30, hep-th/9410167.
11. E. Witten, *String Theory Dynamics In Various Dimensions*, Princeton preprint, hep-th/9503124; C. Vafa, unpublished, as cited therein.
12. A. Strominger, *Massless Black Holes and Conifolds in String Theory*, ITP St. Barbara preprint, hep-th/9504090.
13. B. Greene, D. Morrison and A. Strominger, *Black Hole Condensation and the Unification of String Vacua*, preprint CLNS-95-1335, hep-th/9504145.
14. P. Aspinwall and D. Morrison, *U-Duality and Integral Structures*, preprint CLNS-95-1334, hep-th/9505025.
15. S. Kachru and C. Vafa, *Exact Results for N=2 Compactifications of Heterotic Strings*, preprint HUTP-95/A016, hep-th/9505105.

16. S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, *Second-Quantized Mirror Symmetry*, preprint EFI-95-26, `hep-th/9505162`.

17. C. Montonen and D. Olive, *Phys. Lett.* **72** (1977) 117.

18. D. Olive and E. Witten, *Phys. Lett.* **78** (1978) 97.

19. G. 't Hooft, *Nucl. Phys.* **B190** (1981) 455.

20. See e.g., V. Arnold, A. Gusein-Zade and A. Varchenko, *Singularities of Differentiable Maps I, II*, Birkhäuser 1985.

21. W. Lerche and N.P. Warner, *Nucl. Phys.* **B358** (1991) 571.

22. R. Dijkgraaf, E. Verlinde and H. Verlinde, *Nucl. Phys.* **B352** (1991) 59.