Algebraic and Geometric Structures in String Backgrounds

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ABSTRACT. We give a brief introduction to the study of the algebraic structures – and their geometrical interpretations – which arise in the BRST construction of a conformal string background. Starting from the chiral algebra $\mathcal{A}$ of a string background, we consider a number of elementary but universal operations on the chiral algebra. From these operations we deduce a certain fundamental odd Poisson structure, known as a Gerstenhaber algebra, on the BRST cohomology of $\mathcal{A}$. For the 2D string background, the correponding G-algebra can be partially described in term of a geometrical G-algebra of the affine plane $\mathbf{C}^2$. This paper will appear in the proceedings of *Strings 95*.

1 Introduction

The BRST formalism is a widely, if not universally, recognized approach to the imposition of the Virasoro constraints in string theory (for some early works, see [15][9][33][6][7]). Over the last dozen years physicists and mathematicians alike have pondered the BRST-structure of string backgrounds, both abstract and concrete. (For further discussion of the BRST formalism in string and string field theory, see the paper in this volume by Zwiebach.) During the same period, conformal field theory techniques have played an

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ever increasing role in the theory of string backgrounds (see [2][26].) For a long time, the general theory has been dominated by the standard construction of ghost number one BRST ivariant fields from dimension one primary matter fields (see for example [9]). It is well known that such standard invariant fields form a Lie algebra, at least modulo exact fields.

A number of research groups have understood that the operator product expansion for the background chiral algebra leads to a much richer algebraic structure in the *full* BRST cohomology, where all ghost numbers are on an equal footing [31][34][22][29][14]. In [22], the authors have recognized that the full BRST cohomology of a conformal string background has the structure of a Batalin-Vilkovisky algebra, which is a special type of a Gerstenhaber algebra, or G-algebra. A G-algebra is a generalization of a Poisson algebra, which incorporates simultaneously the structure of a commutative algebra and a Lie algebra [10][11][12]. (For a further discussion of BV- and G-algebras in the context of W-string backgrounds, see the paper in this volume by McCarthy et al.)

The particular G-algebra of the 2D string background is especially complex and has probably made no previous appearance in pure mathematics. In particular, in ghost number one BRST cohomology we obtain a (noncentral) extension of the Lie algebra of vector fields in the plane by an infinite dimensional abelian Lie algebra [22]. In this sense, string theory has enriched our understanding of algebraic structures. At the same time, there is a connection between the G-algebra for 2D strings and the anti-bracket formalism. The latter appears in both the work of Schouten-Nijenhuis [27] on the tensor calculus as well as the work of Batalin-Vilkovisky [1] [30] on the quantization of constrained field theories. Thus, string theory has also illuminated our understanding of geometrical structures.

As a biproduct of our structural analysis of the 2D string background, we obtain a new and very simple description of an explicit basis for the (chiral) BRST cohomology. This basis appears already in the work of Witten and Zwiebach [34]. Our enumeration of the basis exploits the full strength of the G-algebra structure, as well as a beautiful representation of the group $SL(2, \mathbf{C})$ in the cohomology. We predict that the fruitful interaction between mathematics and string theory will continue to be as exciting in the future as it has been over the last twenty-six years.

A note to our mathematical friends: In section three below we use the standard contour integral formalism of conformal field theory. This can be understood algebraically as follows: assume that the chiral algebra $\mathcal{A}$ is actually a commutative quantum operator algebra in the sense of [20]. There are an infinite number of bilinear operations in $\mathcal{A}$ – the circle products. The relationship between the circle products and contour integrals is very simple:

$$\oint_{C_w} \mathcal{O}_1(z)\mathcal{O}_2(w)(z-w)^n dz = \mathcal{O}_1(w) \circ_n \mathcal{O}_2(w). \tag{1.1}$$

With this translation, it is now possible to read the current paper as a follow-up to our paper [20] on CQOAs. For the closely related point of view of vertex operator algebras, see [3][8].

In this paper, we begin with a brief review of the BRST contruction of conformal string backgrounds. The 2D string background will be a fundamental example throughout our discussion. Using the elementary notions such as the normal ordered product and the descent operation, we show how to construct some fundamental algebraic structures on the BRST cohomology. In the case of the 2D string background, these structures can be described to a large extent in terms of the geometric G-algebra of the affine plane $\mathbf{C}^2$. We also describe an explicit basis for the BRST cohomology using the $SL(2,\mathbf{C})$ group action on $\mathbf{C}^2$.

2 BRST Formalism

The chiral algebra of a conformal string background is of the form

$$\mathcal{A} = \mathcal{A}^{ghost} \otimes \mathcal{A}^{matter} \tag{2.2}$$

where $\mathcal{A}^{ghost}$, $\mathcal{A}^{matter}$ are respectively the chiral algebras of the ghost and the matter sectors. Thus explicitly an element $\mathcal{O}$ of $\mathcal{A}$ is a finite sum of holomorphic fields of the form $P(b, \partial b, \cdots, c, \partial c, \cdots)\Phi$, where P is a normal ordered differential polynomial of the ghost fields b, c; Φ is a field in the matter sector. The respective (holomorphic part of the) stress energy tensors of the two sectors are L^{ghost}, L^{matter} whose central charges are respectively $c = 26$, and $c = -26$. The BRST current is

$$J = c(L^{matter} + \tfrac{1}{2}L^{ghost}), \tag{2.3}$$

and the ghost number current is

$$F = cb. \tag{2.4}$$

Let's recall a few basics of the BRST formalism. The BRST charge

$$Q = \oint_{C_0} J(z)dz \tag{2.5}$$

has the property that $[Q,Q] = 2Q^2 = 0$, where square brackets $[\cdot,\cdot]$ denotes the super commutator. Here C_0 is a small contour around 0. A BRST invariant field $\mathcal{O}$ is one which satisfies $[Q,\mathcal{O}\,]=0$; and a BRST exact field is one which is of the form $\mathcal{O}\,=[Q,\mathcal{O}\,']$ for some field $\mathcal{O}\,'$. The BRST cohomology of the string background $\mathcal{A}$ is the quotient space

$$H^*(\mathcal{A}\,)=\{Q\text{-invariant fields}\}/\{Q\text{-exact fields}\} \tag{2.6}$$

which is graded by the ghost number $*$.

A standard way to obtain BRST invariants is as follows. Let V be a primary field of dimension one in $\mathcal{A}^{\,matter}$. Then both cV and $c\partial cV$ are BRST invariants of ghost number 1,2 respectively. The fields $\mathcal{O}\,=1$ and $\mathcal{O}\,=c\partial c\partial^2 c$ are two universal BRST invariants known from the bosonic string theory.

2.1 2D String Background

The chiral algebra $\mathcal{A}_{\,2D}$ of the 2D string background is generated by the fields b, c, ∂X, $\partial\phi$, $e^{\pm(\pm iX-\phi)/\sqrt{2}}$, where X,ϕ are free bosons with the usual OPEs. Their corresponding stress energy tensors are:

$$\begin{aligned} L^X &= -\tfrac{1}{2}(\partial X)^2 \\ L^\phi &= -\tfrac{1}{2}(\partial\phi)^2+\sqrt{2}\partial^2\phi \end{aligned} \tag{2.7}$$

whose respective central charges are $c=1, c=25$.

We now describe some BRST invariants of the chiral algebra $\mathcal{A}_{\,2D}$. Since $e^{(\pm iX+\phi)/\sqrt{2}}$ are matter primary fields of dimension one, we immediately obtain some standard BRST invariants $-ce^{(\pm iX+\phi)/\sqrt{2}}$. We denote them by $Y^+_{1/2,\pm1/2}$. There are also *exotic* BRST invariants which cannot be obtained in the above standard way. For example, in ghost number zero we have

$$\mathcal{O}_{\,1/2,\pm1/2}=\left(cb+\frac{i}{\sqrt{2}}(\pm\partial X-i\partial\phi)\right)e^{(\pm iX-\phi)/\sqrt{2}}. \tag{2.8}$$

These BRST invariants were identified in [23] (see also [4]), and their explicit formulas were given in [31][34]. Explicit formulas for infinitely many BRST invariants of the 2D string background are also known. See [4][25][35][34].

3 Fundamental Operations

Using the antighost field b and contour integral, one can define a number of useful operations on the fields $\mathcal{O}$ in a string background. Given a dimension h field $\mathcal{O}$, we attach to it a field of dimension $h+1$

$$\mathcal{O}^{(1)}(w) = \oint_{C_w} b(z)\mathcal{O}(w)dz \tag{3.9}$$

where C_w is a small coutour around w. We call this linear operation, which reduces ghost number by one, the descent operation. Similarly, to $\mathcal{O}$ we can attach a field of dimension h

$$\Delta\mathcal{O}(w) = \oint_{C_w} b(z)\mathcal{O}(w)(z-w)dz. \tag{3.10}$$

This linear operation is called the Delta operation.

There are two important bilinear operations defined as follows. The first one is the dot product (a.k.a. the normal ordered product). Given two fields $\mathcal{O}_1, \mathcal{O}_2$, we define

$$\mathcal{O}_1(w)\cdot\mathcal{O}_2(w) = \oint_{C_w} \mathcal{O}_1(z)\mathcal{O}_2(w)(z-w)^{-1}dz. \tag{3.11}$$

Under the dot product, both the conformal dimension and the ghost number are additive.

We define also the bracket operation

$$\{\mathcal{O}_1(w), \mathcal{O}_2(w)\} = \oint_{C_w} \mathcal{O}_1^{(1)}(z)\mathcal{O}_2(w)dz. \tag{3.12}$$

Note that conformal dimension is additive, while ghost number is shifted by -1 under the bracket operation. This operation was implicit in [31][34], and was studied in general in [22].

Let

$$\begin{aligned} Q*\mathcal{O}(w) &= \oint_{C(w)} J(z)\mathcal{O}(w)dz = [Q, \mathcal{O}(w)] \\ \Sigma*\mathcal{O}(w) &= \oint_{C(w)} L^{total}(z)\mathcal{O}(w)(z-w)dz. \end{aligned} \tag{3.13}$$

Then we have the following algebra of operations:

$$\begin{aligned} \partial\mathcal{O} &= [Q, \mathcal{O}^{(1)}] \\ \Delta^2 &= 0 \\ (Q*)^2 &= 0 \\ [Q*, \Delta] &= \Sigma* \\ \Sigma*\mathcal{O} &= n\mathcal{O} \text{ iff the conformal dimension of } \mathcal{O} \text{ is } n. \end{aligned} \tag{3.14}$$

Moreover, we have the following identities [22]:

$$\begin{aligned} &(a)\quad Q*(\mathcal{O}_1\cdot\mathcal{O}_2) = (Q*\mathcal{O}_1)\cdot\mathcal{O}_2 + (-1)^{|\mathcal{O}_1|}\mathcal{O}_1\cdot(Q*\mathcal{O}_2) \\ &(b)\quad Q*\{\mathcal{O}_1, \mathcal{O}_2\} = \{Q*\mathcal{O}_1, \mathcal{O}_2\} + (-1)^{|\mathcal{O}_1|-1}\{\mathcal{O}_1, Q*\mathcal{O}_2\} \\ &(c)\quad (-1)^{|\mathcal{O}_1|}\{\mathcal{O}_1, \mathcal{O}_2\} = \Delta(\mathcal{O}_1\cdot\mathcal{O}_2) - (\Delta\mathcal{O}_1)\cdot\mathcal{O}_2 - (-1)^{|\mathcal{O}_1|}\mathcal{O}_1\cdot\Delta\mathcal{O}_2 \end{aligned} \tag{3.15}$$

where $|\mathcal{O}|$ denotes the ghost number of $\mathcal{O}$. Note that the first equation in (3.14) is known as the descent equation [34].

3.1 Induced algebraic structures in BRST cohomology

Let $[\mathcal{O}], [\mathcal{O}_1], [\mathcal{O}_2]$ be cohomology classes in $H^*(\mathcal{A})$. By virtue of the identities (3.14) and (3.15), the above operations induce the following well-defined operations on cohomology:

$$\begin{aligned} \Delta[\mathcal{O}] &= [\Delta\mathcal{O}_{[0]}] \\ [\mathcal{O}_1]\cdot[\mathcal{O}_2] &= [\mathcal{O}_1\cdot\mathcal{O}_2] \\ \{[\mathcal{O}_1],[\mathcal{O}_2]\} &= [\{\mathcal{O}_1,\mathcal{O}_2\}]. \end{aligned} \tag{3.16}$$

Here $\mathcal{O}_{[0]}$ is the projection of $\mathcal{O}$ onto the subspace of $\mathcal{A}$ consisting of fields of zero conformal dimension. For example, given the standard BRST cohomology classes $\mathcal{O}_i = cV_i$, $i = 1, 2$, where the V_i are matter primary fields of conformal dimension one, we have

$$\{[\mathcal{O}_1],[\mathcal{O}_2]\} = [\mathcal{O}_3] \tag{3.17}$$

where

$$\mathcal{O}_3(w) = c(w)\oint_{C_w} V_1(z)V_2(w)dz. \tag{3.18}$$

3.2 Chiral ground ring

Since ghost number is additive under dot product in cohomology, it follows that $H^0(\mathcal{A})$ is closed under this product.

Theorem 3.1 *The dot product in $H^0(\mathcal{A})$ is commutative and associative.*

The commutative associative algebra $H^0(\mathcal{A})$ is called the chiral ground ring of $\mathcal{A}$. For example it is shown in [31][22] that in the case $\mathcal{A} = \mathcal{A}_{2D}$, the ground ring is the polynomial algebra generated by the classes $[\mathcal{O}_{1/2,1/2}]$ and $[\mathcal{O}_{1/2,-1/2}]$.

4 Fundamental Identities

Theorem 4.1 *[22] Let u, v, t be classes in $H^*(\mathcal{A}\ )$ of ghost numbers $|u|, |v|, |t|$ respectively. Then we have*
(a) $u \cdot v = (-1)^{|u||v|} v \cdot u$
(b) $(u \cdot v) \cdot t = u \cdot (v \cdot t)$
(c) $\{u, v\} = -(-1)^{(|u|-1)(|v|-1)} \{v, u\}$
(d) $(-1)^{(|u|-1)(|t|-1)} \{u, \{v, t\}\} + (-1)^{(|t|-1)(|v|-1)} \{t, \{u, v\}\} + (-1)^{(|v|-1)(|u|-1)} \{v, \{t, u\}\} = 0$
(e) $\{u, v \cdot t\} = \{u, v\} \cdot t + (-1)^{(|u|-1)|v|} v \cdot \{u, t\}$

Definition 4.2 *A G-algebra is an integrally graded vector space A^* with dot and bracket products satisfying $|u \cdot v| = |u| + |v|$, $|\{u, v\}| = |u| + |v| - 1$ and the five identities (a) through (e).*

For references on the mathematical theory of G-algebras, see [10][11][12][18].

Proposition 4.3 *The BRST cohomology $H^*(\mathcal{A}\ )$ of a string background $\mathcal{A}$ satisfies further fundamental identities:*
(f) $(-1)^{|u|} \{u, v\} = \Delta(u \cdot v) - (\Delta u) \cdot v - (-1)^{|u|} u \cdot \Delta v$
(g) $|\Delta| = -1$ *and* $\Delta^2 = 0$.

Definition 4.4 *A BV algebra is a G-algebra, A^*, equipped with a linear operation Δ satisfying (f) and (g).*

For references on the mathematical theory of BV algebras, see [19][18][30][32]. For related papers on algebraic structure in BRST cohomology, see [7][13][14][16][17][21] [24][28][29][35].

5 G-algebra of the 2D String Background

An important example of a G-algebra is the one given by the BRST cohomology of the 2D string background:

Theorem 5.1 *As a G-algebra, $H^*(\mathcal{A}_{2D})$ is generated by the four classes $[\mathcal{O}_{1/2,\pm 1/2}]$, $[Y^+_{1/2,\pm 1/2}]$. Moreover, $H^p(\mathcal{A}_{2D})$ vanishes except for $p = 0, 1, 2, 3$.*

Note that $\{[Y^+_{1/2,-1/2}], [Y^+_{1/2,1/2}]\} = [ce^{\sqrt{2}\phi}] \neq 0$. (See [22].)

We now describe the structure of the G-algebra $H^*(\mathcal{A}_{2D})$ in terms of a well-known geometrical G-algebra. Let M be a smooth manifold. Then the space $V^*(M)$ of polyvector fields (ie. antisymmetric contravariant tensor fields) on M admits the structure of a G-algebra, known as the Schouten algebra of M. The dot product of a p-vector field P and a q-vector Q is given by

$$(P \cdot Q)^{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_q} = P^{[\nu_1 \cdots \nu_p} Q^{\mu_1 \cdots \mu_q]}. \tag{5.19}$$

Now let P, Q be elements of $V^{p+1}(M), V^{q+1}(M)$ respectively. The Schouten bracket can be described as follows: in local coordinates the Schouten bracket $[P, Q]_S$ is given by [27]

$$[P, Q]_S^{\nu_1 \cdots \nu_p \lambda \mu_1 \cdots \mu_q} = (p+1) P^{\rho[\nu_1 \cdots \nu_p} \partial_\rho Q^{\lambda \mu_1 \cdots \mu_q]} - (q+1) Q^{\rho[\mu_1 \cdots \mu_q} \partial_\rho P^{\lambda \nu_1 \cdots \nu_p]}. \tag{5.20}$$

Note that if $p = q = 0$, then $[P, Q]_S$ is the ordinary Lie bracket of the vector fields P, Q.

Let $A^*(\mathbf{C}^2)$ be the holomorphic polyvector fields on $\mathbf{C}^2$ with polynomial coefficients. As a linear space, $A^*(\mathbf{C}^2)$ is the super polynomial algebra $\mathbf{C}[x, y, \partial_x, \partial_y]$, where x, y have ghost number zero and ∂_x, ∂_y have ghost number one.

Theorem 5.2 *The assignment $[\mathcal{O}_{1/2,1/2}] \mapsto x$, $[\mathcal{O}_{1/2,-1/2}] \mapsto y$, $[Y^+_{1/2,-1/2}] \mapsto \partial_x$, $[Y^+_{1/2,1/2}] \mapsto \partial_y$ extends to a G-algebra homomorphism ψ of $H^*(\mathcal{A}_{2D})$ onto $A^*(\mathbf{C}^2)$.*

For details on this result, see [22].

Introduce the notations $x^* = \partial_x$, $y^* = \partial_y$. Define the linear operation on $A^*(\mathbf{C}^2)$: $D = \frac{\partial}{\partial x}\frac{\partial}{\partial x^*} + \frac{\partial}{\partial y}\frac{\partial}{\partial y^*}$.

Proposition 5.3 *(a) $A^*(\mathbf{C}^2)$ equipped with D is a BV algebra.*
(b) For u in $H^*(\mathcal{A}_{2D})$, $\psi(\Delta u) = -D(\psi u)$.

(See [22][34].)

Introduce the notations $\tilde{x} = [\mathcal{O}_{1/2,1/2}]$, $\tilde{y} = [\mathcal{O}_{1/2,-1/2}]$, $\tilde{\partial}_x = [Y^+_{1/2,-1/2}]$, $\tilde{\partial}_y = [Y^+_{1/2,1/2}]$, $\tilde{J}_+ = \tilde{x}\tilde{\partial}_y$, $\tilde{J}_0 = \tilde{x}\tilde{\partial}_x - \tilde{y}\tilde{\partial}_y$, $\tilde{J}_- = \tilde{y}\tilde{\partial}_x$. Similarly let $J_+, J_0, J_- \in A^*(\mathbf{C}^2)$ be defined by analogous formulas but without the tildes.

Proposition 5.4 *$Span\{\tilde{J}_+, \tilde{J}_0, \tilde{J}_-\}$, $Span\{J_+, J_0, J_-\}$ are both closed under the bracket $\{\cdot,\cdot\}$ and are isomorphic to the Lie algebra $sl(2, \mathbf{C})$.*

(See [31][34][22][4].)

Now introduce an $sl(2, \mathbf{C})$-action on $H^*(\mathcal{A}_{2D})$ by $\tilde{J}_a * \mathcal{O} = \{\tilde{J}_a, \mathcal{O}\}$ where $a = +, 0, -$, $\mathcal{O} \in H^*(\mathcal{A}_{2D})$. Similarly, introduce an $sl(2, \mathbf{C})$-action on $A^*(\mathbf{C}^2)$ by $J_a * X = \{J_a, X\}$, $X \in A^*(\mathbf{C}^2)$. We make the observation (see [22][4]) that $\psi : H^*(\mathcal{A}_{2D}) \to A^*(\mathbf{C}^2)$ intertwines actions of $sl(2, \mathbf{C})$.

Proposition 5.5 *[22] $ker\ \psi$ is a G-ideal with vanishing dot product and bracket product.*

That is, for $\mathcal{O} \in H^*(\mathcal{A}_{2D})$, $\mathcal{O}', \mathcal{O}'' \in ker\ \psi$, we have $\mathcal{O} \cdot \mathcal{O}', \{\mathcal{O}, \mathcal{O}'\} \in ker\ \psi$, and $\mathcal{O}' \cdot \mathcal{O}'' = \{\mathcal{O}', \mathcal{O}''\} = 0$.

Proposition 5.6 *[22] $ker\ \psi$ as a G-ideal is generated by the class $[ce^{\sqrt{2}\phi}]$.*

This means that $[ce^{\sqrt{2}\phi}] \in ker\ \psi$, and the smallest subspace containing $[ce^{\sqrt{2}\phi}]$ and stable under the action of $H^*(\mathcal{A}_{2D})$ by both the dot product and the bracket product is the whole $ker\ \psi$ itself.

Let's give an explicit basis for $ker\ \psi$. Introduce the following linear operations on cohomology classes $[\mathcal{O}]$:

$$\begin{aligned} A[\mathcal{O}] &= \{\tilde{\partial}_x, [\mathcal{O}]\} \\ B[\mathcal{O}] &= \{\tilde{\partial}_y, [\mathcal{O}]\} \\ C[\mathcal{O}] &= \tilde{\partial}_x \cdot [\mathcal{O}] \\ D[\mathcal{O}] &= \tilde{\partial}_y \cdot [\mathcal{O}]. \end{aligned} \tag{5.21}$$

Fix $\mathcal{K} = ce^{\sqrt{2}\phi}$. Then we have

Proposition 5.7 *$ker\ \psi$ as a vector space has a basis consisting of the following classes:*

$$\begin{aligned} &ghost\ no.\ 1: \quad A^{s-n}B^{s+n}[\mathcal{K}] \\ &ghost\ no.\ 2: \quad (s-n)A^{s-n-1}B^{s+n}C[\mathcal{K}] + (s+n)A^{s-n}B^{s+n-1}D[\mathcal{K}] \\ &ghost\ no.\ 2: \quad A^{s-n+1}B^{s+n}D[\mathcal{K}] - A^{s-n}B^{s+n+1}C[\mathcal{K}] \\ &ghost\ no.\ 3: \quad A^{s-n}B^{s+n}CD[\mathcal{K}] \end{aligned} \tag{5.22}$$

where $s = 0, \frac{1}{2}, 1, ...,$ and $n = -s, -s+1, .., s$.

We observe that the above basis vectors are in fact weight vectors for the $sl(2,\mathbf{C})$ action which we have previously described. Here s, n are respectively the total spin and axial spin quantum numbers of the $sl(2,\mathbf{C})$ representation.

For completeness, we list here a basis for the algebra $A^*(\mathbf{C}^2) \cong H^*(\mathcal{A}_{2D})/ker\ \psi$:

$$\begin{aligned}
&ghost\ no.\ 0: && x^{s-n} \cdot y^{s+n} \\
&ghost\ no.\ 1: && \partial_x(x^{s-n} \cdot y^{s+n}) \cdot \partial_y - \partial_y(x^{s-n} \cdot y^{s+n}) \cdot \partial_x \\
&ghost\ no.\ 1: && x^{s-n} \cdot y^{s+n} \cdot (x \cdot \partial_x + y \cdot \partial_y) \\
&ghost\ no.\ 2: && x^{s-n} \cdot y^{s+n} \cdot \partial_x \cdot \partial_y
\end{aligned} \tag{5.23}$$

where $s = 0, \frac{1}{2}, 1, ...,$ and $n = -s, -s+1, .., s$.

6 Conclusion

Already in [31], Witten begins to give a geometrical interpretation of the BRST cohomology of the 2D string background. He shows that the ghost number zero chiral BRST cohomology is isomorphic to the algebra of holomorphic polynomials on complex 2-space (see section 5). Witten and Zwiebach note that under the dot product, the BRST classes $\tilde{x}$, $\tilde{y}$, $\tilde{\partial}_x$, $\tilde{\partial}_y$ generate a graded commutative associative algebra $H(+)$ isomorphic to $A^*(\mathbf{C}^2)$, which is interpreted geometrically as holomorphic polynomial polyvector fields on complex 2-space. However, since $H(+)$ is clearly not closed under the bracket operation, it is more difficult to give a purely geometrical meaning to that operation. It is also tricky to give geometical meaning to $ker\ \psi$, which constitutes the other "half" of the BRST cohomology.

The point of view taken in our earlier paper [22] as well as in this lecture is that the homomorphism ψ provides us with a geometrical interpretation of the quotient G-algebra, $H^*(\mathcal{A}_{2D})/ker\ \psi$. For example, the $sl(2,\mathbf{C})$ action on the quotient can now be thought of as the infinitesimal counterpart to the geometrical action of the Lie group $SL(2,\mathbf{C})$ on $A^*(\mathbf{C}^2)$. It is interesting already to consider the orbit structure of $SL(2,\mathbf{C})$ on $\mathbf{C}^2$: there are two orbits, neither of which is free: the first consists of the origin only, and the second consists of all nonzero points. Observe that $\mathbf{C}^2 - \{0\} = SL(2,\mathbf{C})/N$ where N is the subgroup of upper triangular unipotent matrices.

For any complex semisimple Lie group G, we can consider an analog of the open orbit of $SL(2,\mathbf{C})$ in $\mathbf{C}^2$: let the N be a maximal unipotent subgroup of G; that is, think of G as a group of complex matrices (we can always do this), and let N be maximal among subroups of G that consist of matrices of the form $I + a\ nilpotent\ matrix$. The coset space G/N is called in mathematics the base affine space of G. In the lecture by McCarthy you will see a far reaching generalization of our 2D string background in which a simple group G of ADE type replaces the $SL(2, \mathbf{C})$ that appears in the theory of the 2D string background.

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