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THE NEW PHYSICS OF STRONGLY COUPLED 2D GRAVITY

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ABSTRACT

The strong coupling physics of two dimensional gravity at $C = 7$, 13, 19 is summarized. It is based on a new set of local fields which do not preserve chirality. Thus this quantum number becomes "deconfined" in the strongly coupled regime. This new set leads to a novel definition of the area elements, and hence to a modified expression for the string suceptibility, which is the real part of the KPZ formula. It allows to define topological (strongly coupled) Liouville string theories (without transverse degree of freedom) which are completely solvable, and are natural extensions of the standard matrix models.

1. Introduction

The point of this seminar is to show, following refs.[1,2,3,4] how the operator approach to Liouville theory remains applicable to the strong coupling regime. Although the Liouville exponentials loose meaning in the strongly coupled regime (since their operator product algebra involves operators and/or highest weight states with complex Virasoro weights), the general operator-family of their chiral components may still be used, when truncation theorems[5,1,2] apply, that is with central charges $C = 7$, 13, 19. Indeed for these values there exist subfamilies of the above chiral operators which form closed operator algebras, and are compatible with the reality condition of Virasoro weights. In the present operator approach they are used to construct a new set of local fields which replace the Liouville exponentials. Since both sets are constructed out of the same free Bäcklund fields, they may be considered as related by a new type of quantum Bäcklund transformation, that connect the weak and strong coupling regimes of two-dimensional gravity. In the present lecture notes, I summarise the basic features (deconfinement of chirality, new expression for the string susceptibility) of these new set of local fields that replace the Liouville exponentials, in the strong coupling regime. Moreover, the basic properties of the new topological models will be recalled, where the gravity part is in the strong coupling regime. The message will be that the derivation of the new features is very close to the previous weak-coupling one, once the new set of local physical fields is established.

2. The weak coupling regime revisited

Remarkably, the present operator method treats the weak and strong coupling regimes much on the same footing. As a preparation, let us recall some basic points of the weak-coupling discussion[7,8,9]. Our conventions are to let

$$Q_L=\sqrt{(C-1)/3}, \quad Q_M=\sqrt{(25-C)/3} \tag{1}$$

where C is the Liouville central charge. We call ϑ_L, and $\overline{\vartheta}_L$ the two chiral components of the Bäcklund free field associated with the Liouville field Φ. The building blocks of the quantum group approach to Liouville theory are chiral fields of the form

$$\widetilde{V}_{m\widehat{m}}^{(J\,\widehat{J})} \propto V_{-J\,-\widehat{J}}^{(J\,\widehat{J})} S^{J+m}\,\widehat{S}^{\widehat{J}+\widehat{m}}, \quad \widetilde{\overline{V}}_{\overline{m}\,\widehat{\overline{m}}}^{(\overline{J}\,\widehat{\overline{J}})} \propto \overline{V}_{-\overline{J}\,-\widehat{\overline{J}}}^{(\overline{J}\,\widehat{\overline{J}})} \,\overline{S}^{\overline{J}+\overline{m}}\,\widehat{\overline{S}}^{\widehat{\overline{J}}+\widehat{\overline{m}}}, \tag{2}$$

The V fields are functions of z, and the $\overline{V}$ fields functions of $\bar{z}$. The fields $V_{-J-\widehat{J}}^{(J\widehat{J})}$ are simple exponentials which may be directly re-expressed in terms of the Bäcklund free fields.

$$V_{-J-\widehat{J}}^{(J\widehat{J})} = :\exp\Big[(J\alpha_- + \widehat{J}\alpha_+)\vartheta_L\Big]:, \quad \overline{V}_{-\overline{J}\,-\widehat{\overline{J}}}^{(\overline{J}\widehat{\overline{J}})} = :\exp\Big[(\overline{J}\alpha_- + \widehat{\overline{J}}\alpha_+)\overline{\vartheta}_L\Big]: . \tag{3}$$

S, $\overline{S}$, and $\widehat{S}$, $\widehat{\overline{S}}$ are the screening operators[8] associated with the two screening charges

$$\alpha_\pm = Q_L/2 \pm iQ_M/2. \tag{4}$$

The J's are quantum group spins, but we will not dwell upon this aspect. In practice they determine the weight of the V operators which are of the type $(2\widehat{J}+1, 2J+1)$ in the BPZ classification. We deal with irrational theories, since this will be the case in the strong coupling regime. Thus the range of J and $\widehat{J}$ is unbounded. Concerning the Hilbert space, the Verma modules is of course charaterized by the same four quantum group spins. The highest weight states will be noted $|J,\,\widehat{J}>|\overline{J},\,\widehat{\overline{J}}>$. Following the earlier studies[9], the chiral operators just written are noted with a tilde to emphasize their special normalization which is such that their fusing and braiding matrices are exactly equal to q-6j symbols. The corresponding highest-weight matrix elements of the chiral fields define the coupling constants[9,2].

The local Liouville exponentials are given by[1,8]

$$e^{-(J\alpha_- + \widehat{J}\alpha_+)\Phi(z,\bar{z})} = \sum_{m,\,\widehat{m}} \widetilde{V}_{m\,\widehat{m}}^{(J,\widehat{J})}(z)\,\widetilde{\overline{V}}_{m\,\widehat{m}}^{(J,\widehat{J})}(\bar{z}) \tag{5}$$

This form is dictated by locality and closure under fusion. For the following, it is important to stress that this expression only involves V and $\overline{V}$ fields with equal quantum numbers ($J = \overline{J}$, $m = \overline{m}$, and so on), while m, amd $\widehat{m}$ are summed over independently. Thus the Liouville exponentials have zero conformal spins. On the other hand, it follows from the formulae just summarized that

$$< J_2,\ \widehat{J}_2 | \widetilde{V}^{(J\widehat{J})}_{m\widehat{m}} | J_1,\ \widehat{J}_1 > \propto \delta_{J_1-J_2-m,0}\, \delta_{\widehat{J}_1-\widehat{J}_2-\widehat{m},0}, \tag{6}$$

so that the Liouville exponential applied to a highest-weight state with $J = \overline{J}$, $\widehat{J} = \overline{\widehat{J}}$ only gives states satisfying the same condition. Thus we may restrict ourselves to the subsector with zero winding number. We stress this well known fact, since this will not be true any more in the strong coupling regime. Let us next recall some main point of the derivation of the matrix model results in the present context following refs.[7,2]. For $C > 25$, Q_L is real and Q_M pure imaginary, so that $\alpha_\pm$ are real. The above formulae are directly useful. One represents matter by another copy of the theory summarized above, now with central charge $c = 26 - C$, so that Q_M is its background charge. One constructs local fields in analogy with Eq.5:

$$e^{-(J\alpha'_- + \widehat{J}\alpha'_+)\Phi'(z,\bar{z})} = \sum_{m,\,\widehat{m}} \widetilde{V}'^{\,(J,\widehat{J})}_{m\,\widehat{m}}(z)\, \overline{\widetilde{V}}'^{\,(J,\widehat{J})}_{m\,\widehat{m}}(\bar{z}). \tag{7}$$

Symbols pertaining to matter are distinguished by a prime (or, if more convenient by the index M). In particular $\Phi'(z,\bar{z})$ is the matter field (it commutes with $\Phi(z,\bar{z})$), and $\alpha'_\pm$ are the matter screening charges

$$\alpha'_\pm = \mp i \alpha_\mp. \tag{8}$$

The correct dressing of these operators by gravity is achieved by considering the vertex operators

$$\mathcal{W}^{J,\widehat{J}} \equiv e^{-((-\widehat{J}-1)\alpha_- + J\alpha_+)\Phi \,-\, (J\alpha'_- + \widehat{J}\alpha'_+)\Phi'} \tag{9}$$

which is an operator of weights $\Delta = \bar{\Delta} = 1$. In particular for $J = \widehat{J} = 0$, we get the cosmological term $\exp(\alpha_- \Phi)$. The three-point function was computed in refs[7,2]. The corresponding product of coupling constants gives the correct leg factors after drastic simplifications. After that, one may follow the line of ref.[10] and derive the higher point function. We will come back to this in the coming section.

3. The new local fields.

At this point we turn to the strong coupling regime. Now $1 < C < 25$, and Q_L, Q_M are real. The screening charges $\alpha_\pm$ are complex and related by complex conjugation. Thus complex weights appear in general. There are two types of exceptional cases. The states $|J,\ J>$ (resp. $|-J-1,\ J>$) have highest weights which are real and negative (resp. positive). One could try to work with the corresponding Liouville exponentials $\exp[-J(\alpha_- - \alpha_+)\Phi]$ (resp $\exp[((J+1)\alpha_- - \alpha_+)\Phi]$), but this would be inconsistent, since these operators do not form a closed set under fusing and braiding. Moreover, as is clear from Eqs.5, 6, they do not preserve the reality condition for highest weights just recalled. The basic problem is that Eq.5 involves the V operators with arbitrary $m\, \widehat{m}$, while the reality condition forces us to only use V operators of the type

$$V_{m,+}^{(J)} \equiv \widetilde{V}_{-m\,m}^{(-J-1,J)}, \quad V_{m,-}^{(J)} \equiv \widetilde{V}_{m\,m}^{(J,J)}. \tag{10}$$

Now is a good time to recall the truncation theorems which hold for

$$C = 1 + 6(s+2), \quad s = 0, \pm 1. \tag{11}$$

First define the physical Hilbert space

$$\mathcal{H}_{\text{phys}}^{\pm} \equiv \bigoplus_{r=0}^{1\mp s} \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_{r/2(2\mp s)+n/2}^{\pm} \tag{12}$$

where $\mathcal{H}_J^{\pm}$ denotes the Verma modules with highest weights $|\mp(J+1/2)-1/2,\ J>$. The physical operators $\chi_{\pm}^{(J)}$ are defined for arbitrary[a] $2J \in \mathcal{Z}/(2\mp s)$, and $2J_1 \in \mathcal{Z}/(2\mp s)$. to be such that[b]

$$\chi_{\pm}^{(J)}\, \mathcal{P}_{\mathcal{H}_{J_1}^{\pm}} = \sum_{\nu \equiv J+m \in \mathcal{Z}_+} (-1)^{(2\mp s)(2J_1+\nu(\nu+1)/2)} V_{m,\pm}^{(J)}\, \mathcal{P}_{\mathcal{H}_{J_1}^{\pm}}, \tag{13}$$

where $\mathcal{P}_{\mathcal{H}_{J_1}^{\pm}}$ is the projector on $\mathcal{H}_{J_1}^{\pm}$. Denote by $\mathcal{A}_{\text{phys}}^{\pm}$ the set of fields $\chi_{\pm}^{(J)}$, with $2J \in \mathcal{Z}/(2\mp s)$. The basic properties of the special values Eq.11 is the TRUNCATION THEOREM:

For $C = 1+6(s+2)$, $s = 0, \pm 1$, and when it acts on $\mathcal{H}_{\text{phys}}^{+}$ (resp. $\mathcal{H}_{\text{phys}}^{-}$), the above set $\mathcal{A}_{\text{phys}}^{+}$ (resp. $\mathcal{A}_{\text{phys}}^{-}$) is closed by braiding and fusion and only gives states that belong to $\mathcal{H}_{\text{phys}}^{+}$ (resp. $\mathcal{H}_{\text{phys}}^{-}$).

Note that the operators $V_{m,\pm}^{(J)}$ themselves are not closed by fusing and braiding, contrary to the very specific combinations Eq.13. The proof is given in refs.[1,2]. It follows from a neat mathematical property of the quantum group structure. In general, it is of the type[7] $U_q(sl(2)) \odot U_{\widehat{q}}(sl(2))$ with $h = \pi\alpha_-^2/2$, $\widehat{h} = \pi\alpha_+^2/2$, so that $h\widehat{h} = \pi^2$. The two quantum group parameters are thus related by duality. At the special values, one has, in addition $h + \widehat{h} = s\pi$. Then the q-6j symbols of the two dual quantum groups become equal up to a sign[2]. Using the orthogonality relation of the q-6j's this leads to the truncation theorems.

[a] By the symbol $\mathcal{Z}/(2\pm s)$, we mean the set of numbers $r/(2\pm s)+n$, with $r = 0, \cdots, 1\pm s$, n integer; $\mathcal{Z}$ denotes the set of all positive or negative integers, including zero.
[b] $\mathcal{Z}_+$ denotes the set of non negative integers.

Next we construct local fields out of the chi fields. The braiding of the chi fields is a simple phase. On the unit circle, one has

$$\chi_{\pm}^{(J_1)}\chi_{\pm}^{(J_2)}=e^{2i\pi\epsilon(2\mp s)J_1J_2}\chi_{\pm}^{(J_2)}\chi_{\pm}^{(J_1)}, \tag{14}$$

where $\epsilon=\pm 1$ is fixed by the ordering of the operator on the left-hand side in the usual way. From the spectrum of the J's, it follows that the phase factor is of the form $\exp(i\pi N/2(2\mp s))$, where $N\in\mathcal{Z}$. Thus, we have parafermions. As shown in ref.[2], simple products of the form $\chi_{\pm}^{(J)}\overline{\chi}_{\pm}^{(\overline{J})}$, with $J-\overline{J}\in\mathcal{Z}$ are local. In such a product, the summations over m, and $\overline{m}$ are independent, while the summations over m, $\widehat{m}$, and $\overline{m}$, $\widehat{\overline{m}}$ are correlated. Now we have a complete reversal of the weak coupling situation summarized by Eq.5: the new fields preserve the reality condition, but **do not preserve the equality between J and $\overline{J}$ quantum numbers**. Thus, as already stressed in ref.[2], in the strong coupling regime, we observe a sort of deconfinement of chirality.

4. The Liouville string

One may consider two different problems. First, one may build a full-fledged string theory, by coupling, for instance, the above with $26-C$ free fields $\vec{X}$. A typical string vertex is of the form $\exp(i\vec{k}.\vec{X})\chi_{+}^{(J)}\overline{\chi}_{+}^{(\overline{J})}$, where $\vec{k}$, J, and $\overline{J}$ are related so that this is a $1,1$ operator. Here obviously, the restriction to real weight is instrumental. Moreover, since one wants the representation of Virasoro algebra to be unitary, one only uses the chi+ fields. This line was already persued with noticable success in refs.[6]. However, the N-point functions seem to be beyond reach at present. Second a simpler problem seems to be tractable, namely, we may proceed as in the construction of topological models just recalled. We consider another copy of the present strongly coupled theory, with central charge $c=26-C$. Since this gives $c=1+6(-s+2)$, we are also at the special values, and the truncation theorems applies to matter as well. This "string theory" has no transverse degree of freedom, and is thus topological. The complete dressed vertex operator is now

$$\mathcal{V}^{J,\overline{J}}=\chi_{+}^{(J)}\overline{\chi}_{+}^{(\overline{J})}\,\chi'^{\,(J)}_{-}\overline{\chi}'^{\,(\overline{J})}_{-} \tag{15}$$

As in the weak coupling formula, operators relative to matter are distinguished by a prime. The definition of the $\overline{\chi}$ is similar to the above, with an important difference. Clearly, the definition Eq.10 of $V_{m,+}^{(J)}$ is not symmetric between α_+, and α_-. The truncation theorems also holds if we interchange the two screening charges. We re-establish some symmetry between them by taking the other possible definition for $\overline{\chi}$, namely, we let

$$\overline{V}_{\overline{m},+}^{(\overline{J})}\equiv\widetilde{\overline{V}}_{\overline{m}\,-\overline{m}}^{(\overline{J},\,-\overline{J}-1)},\quad V_{\overline{m},-}^{(\overline{J})}\equiv\widetilde{V}_{\overline{m}\,\overline{m}}^{(\overline{J},\overline{J})}. \tag{16}$$

Our results will then be invariant by complex conjugation provided we exchange J's and $\overline{J}$'s. Thus left and right movers are interchanged, which seems to be a sensible requirement. For $J=\overline{J}=0$, we get the new cosmological term

$$\mathcal{V}^{0,0}=\chi_+^{(0)}(z)\overline{\chi}_+^{(0)}(\bar{z}). \tag{17}$$

Thus the area element of the strong coupling regime is $\chi_+^{(0)}(z)\overline{\chi}_+^{(0)}(\bar{z})dzd\bar{z}$. It is factorized into a simple product of a single z component by a $\bar{z}$ component. From this expression one may compute the string susceptibility using the operator version of the DDK argument developed in ref.[7] for the weak coupling regime. For this we introduce the cosmological constant — so far it was set equal to one. In ref.[7], the weak coupling string susceptibility was rederived from the following ansatz[c]

$$\begin{aligned}
\left.\widetilde{V}_{m\,\widehat{m}}^{(J\,\widehat{J})}\right|_{(\mu)} &= \mu^{J+\widehat{J}\alpha_+/\alpha_-}\mu^{-ip_0/\alpha_-}\,\widetilde{V}_{m\,\widehat{m}}^{(J\,\widehat{J})}\;\mu^{ip_0/\alpha_-}\\
\left.\widetilde{\overline{V}}_{\overline{m}\,\widehat{\overline{m}}}^{(\overline{J}\,\widehat{\overline{J}})}\right|_{(\bar{\mu})} &= \bar{\mu}^{\overline{J}+\widehat{\overline{J}}\alpha_+/\alpha_-}\bar{\mu}^{-i\overline{p}_0/\alpha_-}\,\widetilde{\overline{V}}_{\overline{m}\,\widehat{\overline{m}}}^{(\overline{J}\,\widehat{\overline{J}})}\;\bar{\mu}^{i\overline{p}_0/\alpha_-}
\end{aligned} \tag{18}$$

where p_0 is the zero-mode momentum of the ϑ_L free field. We take a priori two different parameters μ and $\bar{\mu}$. For the weak coupling case, the previous discussion is immediately recovered with $\mu_c=\mu\bar{\mu}$ which is the only parameter that counts. For the strong coupling regime, one substitutes the last equation into Eq.13, and its antichiral couterpart. We have defined the $\overline{\chi}$ fields so that taking hermitian conjugate is equivalent to exchanging J's and $\overline{J}$'s. We shall thus relate μ and $\bar{\mu}$ so that the prefactors are connected in the same way. Taking μ real, this is realized if $\bar{\mu}^{\alpha_+}=\mu^{\alpha_-}$. Then we have

$$\left.\chi_+^{(J)}\right|_{(\mu)}\left.\overline{\chi}_+^{(\overline{J})}\right|_{(\bar{\mu})}=\mu^{-2+Q_M(\alpha'_-J+\alpha'_+\overline{J})/2}\mu^{-i(p_0/\alpha_-+\overline{p}_0/\alpha_+)}\chi_+^{(J)}\overline{\chi}_+^{(\overline{J})}\mu^{i(p_0/\alpha_-+\overline{p}_0/\alpha_+)} \tag{19}$$

For the cosmological term $J=\overline{J}=0$, and the factor becomes μ^{-2}. Thus we conclude that $\mu_c=\mu^2$. Using the operator definition of correlators (see refs.[7,9]), and applying Eq.19, we get

$$\left\langle\prod_\ell \chi_+^{(J_\ell)}\overline{\chi}_+^{(\overline{J}_\ell)}\right\rangle_{(\mu_c)}=\left\langle\prod_\ell \chi_+^{(J_\ell)}\overline{\chi}_+^{(\overline{J}_\ell)}\right\rangle\mu_c^{-\sum_\ell[1-Q_M(\alpha'_-J_\ell+\alpha'_+\overline{J}_\ell)/4]+(s+2)/2}. \tag{20}$$

The last terms emerges when the operators $\mu^{\pm i(p_0/\alpha_-+\overline{p}_0/\alpha_+)}$ hit the left and the right vacuum states (of course due to the background charge). This scaling law is enough to compute[d] the string susceptibility. Consider

$$\mathcal{Z}_{\mu_c}(A)\equiv\left\langle\delta\left[\int dzd\bar{z}\;\left.\chi_+^{(0)}\right|_{(\mu_c)}\left.\overline{\chi}_+^{(0)}\right|_{(\mu_c)}-A\right]\right\rangle. \tag{21}$$

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[c] The previous discussion was actually somewhat different, since V and $\overline{V}$ operators were treated differently. This does not make any difference for the weak coupling regime, but matters at present.
[d] We only consider genus zero.

One gets

$$\gamma_{\rm str} = (2-s)/2. \tag{22}$$

The result is real for $c>1$ ($C<25$), contrary to the continuation of the weak-coupling equation $\gamma_{\rm str}=2-Q/\alpha_-$. Explicitly one has

$$\left\{\begin{array}{cccc} s & c & C & \gamma_{\rm str} \\ 1 & 7 & 19 & 1/2 \\ 0 & 13 & 13 & 1 \\ -1 & 19 & 7 & 3/2 \end{array}\right. , \qquad \left\{\begin{array}{cccc} s & c & C & \gamma_{\rm str} \\ 2 & 1 & 25 & 0 \\ -2 & 25 & 1 & 2 \end{array}\right. \tag{23}$$

The last two are the extreme points of the strong coupling regime. The values at $c=1$, and $c=25$ agree with the weak-coupling formula. The result is always positive, contrary to the weak-coupling regime. At $c=7$, we find the value $\gamma_{\rm str}=1/2$ of branched polymers.

Finally, we have computed the N-point functions with one incoming and N-1 outgoing legs, defined as follows. First in general[9] the two-point function of two $\widetilde{V}$ fields with spins J_1, $\overline{J}_1$, and J_2, $\overline{J}_2$ vanishes unless $J_1+J_2+1=0$, and $\overline{J}+\overline{J}_2+1=0$. Thus conjugation involves the transformation $J\to -J-1$. Taking account of the exchange between J and $\widehat{J}$, due to complex conjugation, yields the following vertex operator for the conjugate representation:

$$\mathcal{V}^{J,\overline{J}}_{conj} = \chi_+^{(J)}\,\overline{\chi}_+^{(\overline{J})}\,\chi'^{\,(-J-1)}_{\,-}\,\overline{\chi}'^{\,(-\overline{J}-1)}_{\,-}. \tag{24}$$

Thus we have computed the matrix elements $\left\langle \mathcal{V}^{J_1,\overline{J}_1}_{conj}\mathcal{V}^{J_2,\overline{J}_2}\ldots\mathcal{V}^{J_N,\overline{J}_N}\right\rangle$. The method is similar to the one developed in ref.[10], with a reshuffling of quantum numbers. In the weak coupling regime, left and right quantum numbers are kept equal, while the ones associated with different screening charge are chosen independently. In the strong coupling regime, the situation is reversed: the reality condition ties the quantum numbers which differ by the screening charge, but the quantum numbers with different chiralities become independent. See refs.[3,4] for details.

5. Hints for future developments.

In the present topological models, both matter and gravity have a background charge. By construction, the stress-energy tensor takes the usual free-field form after Bäcklund transformation. It is thus clearly possible to recombine the Liouville Bäcklund field ϑ_L, with its matter counterpart ϑ_M so that the background charge appears in one of the free fields only. For this we let

$$\vartheta_L = -\vec{X}.\vec{\mu}_L, \quad \vartheta_M = \vec{X}.\vec{\mu}_M. \tag{25}$$

where $\vec{X}\equiv(\varphi, X)$, and we introduce the two orthonormalized vectors

$$\vec{\mu}_L = \left(\frac{Q_L}{2\sqrt{2}}, \quad \frac{Q_M}{2\sqrt{2}}\right), \quad \vec{\mu}_M = \left(\frac{Q_M}{2\sqrt{2}}, \quad -\frac{Q_L}{2\sqrt{2}}\right). \tag{26}$$

Our conventions for $\vec{X}$ coincide with the one of ref.[11], and the present fields are identical **apart from the zero-mode spectrum.** Let $\vec{X}_0$ be the center-of-mass position. It is easy to see that $V^{(J)}_{m,\pm} \propto \exp(im\vec{k}_\pm.\vec{X}_0)$, and $V'^{(J)}_{m,\pm} \propto \exp(im\vec{k}'_\pm.\vec{X}_0)$, where $\vec{k}_- = -iQ_L\vec{\mu}_L$, $k_+ = Q_M\vec{\mu}_L$, $\vec{k}'_- = -iQ_M\vec{\mu}_M$, and $\vec{k}'_+ = Q_L\vec{\mu}_M$. Remembering the condition $J+m \in \mathcal{Z}_+$ of eq.13, we see that these momenta lie on the lattice generated by the vectors

$$\frac{-i}{2\sqrt{2}}\left(1,\sqrt{\frac{2-s}{2+s}}\right), \qquad \frac{1}{2\sqrt{2}}\left(\sqrt{\frac{2+s}{2-s}},\,1\right),$$
$$\frac{-i}{2\sqrt{2}}\left(1,-\sqrt{\frac{2+s}{2-s}}\right), \qquad \frac{1}{2\sqrt{2}}\left(\sqrt{\frac{2-s}{2+s}},\,-1\right), \tag{27}$$

which is itself embedded in a four dimensional space with signature $2,2$. Thus there may exist a connection between our topological theories and the $N=2$ superstring.[e] Since $Q_M\vec{\mu}_L/2 - Q_L\vec{\mu}_M/2 = \left(0,\sqrt{2}\right)$, it follows that the momenta of the $SU(2)$ generators $\exp \pm i\sqrt{2}X$ belong to the above lattice. Moreover, $k_{+X} = (2-s)/\sqrt{2}$, $k'_{+X} = -(2+s)/\sqrt{2}$. Thus, for integer $s = 0, \pm 1$ there are points of the lattice which differ by the momenta $(0, \pm\sqrt{2})$ of the $SU(2)$ generators. Of course, only the $\vec{X}_0$ dependence of the χ fields is simple. In general Eqs.13, and 15 show that $\mathcal{V}^{J,\bar{J}}$ is a rather involved function of $\vec{X}$ involving momenta of the form $m(\vec{k}_+ + \vec{k}'_-)$. Since that $(\vec{k}_+ + \vec{k}'_-)^2 = 0$, all our on-shell string states are massless, and orthogonal to each other.

Let us turn to a final remark. The redefinition of the cosmological term led us to modify the KPZ formula[13]. On the other hand, in standard studies of the matrix models or KP flows, one first derives $\gamma_{\rm str}$ and deduces the value of the central charge by assuming that the KPZ formula holds. In this way of thinking, one would start from our formula Eq.22 and apply KPZ, which would lead to a different value of the central charge, say d. It is easy to see that for $c = 1 + 6(-s+2)$ one gets $d = 1 - 6(2-s)^2/2s$. This is the value of a $2,s$ minimal model! What happens is that in terms of d, we have $\gamma_{\rm str} = (d - 1 + \sqrt{(d-1)(d-25)})/12$, in contrast with the KPZ formula $(d - 1 - \sqrt{(d-1)(d-25)})/12$. Thus the strongly coupled topological theories may be given by another branch of $d<1$ theories.

6. Concluding remarks

We should probably stress that no total conformal spin has been introduced. Indeed, although, the conformal spin is non zero for the gravity, and matter components of our vertex operators separately, the total weight for the left and right components are kept equal to one.

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[e]See, e.g. ref.[12].

One may wonder why the present approach succeeds to break through the $c = 1$ barrier, in sharp contrast with the other ones. This may be traced to the fact that we first deal with the chiral components of gravity and matter separately before reconstructing the vertex operators. This is more painful than the matrix model approach which directly constructs the expectation values of the dressed matter operators. However, in this way we have a handle over the way the gravity quantum numbers are coupled, and so we may build up vertex operators which change the gravity chirality. This seems to be the key to the $c = 1$ problem, since this quantum number plays the role of order parameter.

7. References

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