STRINGS IN COSMOLOGICAL SPACETIMES AND THEIR BACK-REACTION

H. J. DE VEGA

Laboratoire de Physique Théorique et Hautes Energies,
Université Pierre et Marie Curie (Paris VI) et Université Denis Diderot (Paris VII),
Tour 16, 1er. étage, 4, Place Jussieu
75252 Paris, cedex 05, France. Laboratoire Associé au CNRS URA280.

Lecture delivered at STRINGS '95, Future Perspectives in String Theory, USC, Los Angeles, march 13 - 18, 1995.

ABSTRACT

This is a short review on strings in curved spacetimes. We start by recalling the classical and quantum string behaviour in singular plane waves backgrounds. We then report on the string behaviour in cosmological spacetimes (FRW, de Sitter, power inflation) which is by now largerly understood. Recent progress on self-consistent solutions to the Einstein equations for string dominated universes is reviewed. The energy-momentum tensor for a gas of strings is considered as source of the spacetime geometry. The string equation of state is determined from the behaviour of the explicit string solutions. This yields a self-consistent cosmological solution exhibiting realistic matter dominated behaviour $R \sim (T)^{2/3}$ for large times and radiation dominated behaviour $R \sim (T)^{1/2}$ for early times. Inflation in the string theory context is discussed.

1. Strings and Quantum Gravity

The construction of a sensible quantum theory of gravitation is probably the greatest challenge in today's theoretical physics. Deep conceptual problems (as the lost of quantum coherence) arise when one tries to combine (second) quantization concepts with General Relativity. That is, it may be very well that a quantum theory of gravitation needs new concepts and ideas. Of course, this future theory must have the today's General Relativity and Quantum Mechanics (and QFT) as limiting cases. In some sense, what everybody is doing in this domain (including string theories approach) may be to the real theory what the old quantum theory in the 10's was compared with quantum mechanics [1].

The main drawback to develop a quantum theory of gravitation is clearly the total lack of experimental guides for the theoretical development. The physical effects combining gravitation and quantum mechanics are relevant only at energies of the order of $M_{Planck} = \hbar c/G = 1.22\,10^{16}$Tev. Such energies were available in the Universe at times $t < t_{Planck} = 5.4\,10^{-44}$sec. Anyway, as a question of principle, to construct a quantum theory of gravitation is a problem of fundamental relevance for theoretical physics . In addition, one cannot rule out completely the possibility of some "low energy" ($E \ll M_{Planck}$) physical effect that could be experimentally tested.

Since M_{Planck} is the heaviest possible particle scale, a theory valid there (necessarily involving quantum gravitation) will also be valid at any lower energy scale. One may ignore higher energy phenomena in a low energy theory, but not the opposite. In other words, it will be a 'theory of everything'. We think that this is the **key point** on the quantization of gravity. A theory that holds till the Planck scale must describe **all** what happens at lower energies including all known particle physics as well as what we do not know yet (that is, beyond the standard model). Notice that this conclusion is totally independent of the use of string models. A direct important consequence of this conclusion, is that it does not make physical sense to quantize **pure gravity**. A physically sensible quantum theory cannot contain only gravitons. To give an example, a theoretical prediction for graviton-graviton scattering at energies of the order of M_{Planck} must include all particles produced in a real experiment. That is, in practice, all existing particles in nature, since gravity couples to all matter[1].

String theory is a serious candidate for a quantum description of gravity since it provides a unified model of all interactions overcoming at the same time the non-renormalizable character of quantum fields theories of gravity.

As a first step on the understanding of quantum gravitational phenomena in a string framework, we started in 1987 a programme of string quantization on curved spacetimes [3,4].

The investigation of strings in curved spacetimes is currently the best framework to study the physics of gravitation in the context of string theory since it provides essential clues about the physics in this context but is clearly not the end of the story. The next step beyond the investigation of **test** strings, consist in finding **self-consistently** the geometry from the strings as matter sources for the Einstein equations or better the string effective equations (beta functions). This goal is achieved in [5] for cosmological spacetimes at the classical level. Namely, we used the energy-momentum tensor for a gas of strings as source for the Einstein equations and we solved them self-consistently. [For more detailed reviews see [1,2]].

Let us consider bosonic strings (open or closed) propagating in a curved D-dimensional spacetime defined by a metric $G_{AB}(X), 0 \leq A, B \leq D-1$. The action can be written as

$$S = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{g}\, g_{\alpha\beta}(\sigma,\tau)\, G_{AB}(X)\, \partial^\alpha X^A(\sigma,\tau)\, \partial^\beta X^B(\sigma,\tau) \qquad (1)$$

Here $g_{\alpha\beta}(\sigma,\tau)$ ($0 \leq \alpha, \beta \leq 1$) is the metric in the worldsheet, α' stands for the string tension. As in flat spacetime, $\alpha' \sim (M_{Planck})^{-2} \sim (l_{Planck})^2$ fixes the scale in the theory.

We will start considering given gravitational backgrounds $G_{AB}(X)$. That is, we start to investigate *test* strings propagating on a given spacetime. In section 3, the back reaction problem will be studied. That is, how the strings may act as source of the geometry.

String propagation in massless backgrounds other than gravitational (dilaton, antisymmetric tensor) can be investigated analogously.

The string equations of motion and constraints follow by extremizing eq.(1) with respect to $X^A(\sigma,\tau)$ and $g_{\alpha\beta}(\sigma,\tau)$, respectively. In the conformal gauge, they take the form:

$$\partial_{-+}X^A(\sigma,\tau)+\Gamma^A_{BC}(X)\,\partial_+X^B(\sigma,\tau)\,\partial_-X^C(\sigma,\tau)=0\ ,\quad 0\le A\le D-1, \tag{2}$$

$$T_{\pm\pm}\equiv G_{AB}(X)\,\partial_\pm X^A(\sigma,\tau)\,\partial_\pm X^B(\sigma,\tau)=0\,,\quad T_{+-}\equiv T_{-+}\equiv 0 \tag{3}$$

where we introduce light-cone variables $x_\pm \equiv \sigma\pm\tau$ on the world-sheet and where $\Gamma^A_{BC}(X)$ stand for the connections (Christoffel symbols) associated to the metric $G_{AB}(X)$.

Notice that these equations in the conformal gauge are still invariant under the conformal reparametrizations:

$$\sigma+\tau\to\sigma'+\tau'=f(\sigma+\tau)\qquad,\qquad \sigma-\tau\to\sigma'-\tau'=g(\sigma-\tau) \tag{4}$$

Here $f(x)$ and $g(x)$ are arbitrary functions.

The string boundary conditions in curved spacetimes are identical to those in Minkowski spacetime. That is,

$$\begin{aligned} X^A(\sigma+2\pi,\tau) &= X^A(\sigma,\tau) \qquad \text{closed strings}\\ \partial_\sigma X^A(0,\tau) &= \partial_\sigma X^A(\pi,\tau)=0 \qquad \text{open strings.} \end{aligned} \tag{5}$$

We shall consider, as usual, that only four space-time dimensions are uncompactified. That is, we shall consider the strings as living on the tensor product of a curved four dimensional space-time with lorentzian signature and a compact space which is there to cancel the anomalies. Therefore, from now on strings will propagate in the curved (physical) four dimensional space-time. However, we will find instructive to study the case where this curved space-time has dimensionality D, where D may be 2, 3 or arbitrary.

2. Strings Falling into Spacetime Singularities: nonlinear plane waves.

Let us first consider strings propagating in gravitational plane-wave space-times [6,7]. In this geometry the full non-linear string equations (2) and constraints (3) can be exactly solved in closed form. The plane-wave space-times are described by the metric

$$(ds)^2=-dU\,dV+\sum_{i=1}^{D-2}(dX^i)^2-\Big[W_1(U)\,(X^2-Y^2)+2\,W_2(U)\,XY\Big]\,(dU)^2 \tag{6}$$

where $X\equiv X^1$, $Y\equiv X^2$. These space-times are exact solutions of the vacuum Einstein equations for any choice of the profile functions $W_1(U)$ and $W_2(U)$. In addition they are exact string vacua [10]. The case when $W_2(U) = 0$ describes waves of constant polarization. When both $W_1(U) \neq 0$ and $W_2(U) \neq 0$, eq.(6) describes waves with arbitrary polarization. If $W_1(U)$ and/or $W_2(U)$ are singular functions, space-time singularities will be present. The singularities will be located on the null plane $U =$ constant. We consider profiles which are nonzero only on a finite interval $-T < U < T$, and which have power-type singularities [6,7],

$$W_1(U) \overset{U\to 0}{=} \frac{\alpha_1}{|U|^{\beta_1}} \quad , \quad W_2(U) \overset{U\to 0}{=} \frac{\alpha_2}{|U|^{\beta_2}} \tag{7}$$

The spacetimes (6) share many properties with the shockwaves [8,9]. In particular, $U(\sigma,\tau)$ obeys the d'Alembert equation and we can choose the light-cone gauge

$$U = 2\,\alpha' p^U \tau\ . \tag{8}$$

The string equations of motion (2) become then in the metric (6) :

$$\begin{aligned} V'' - \ddot{V} \ + \ (2\alpha' p^U)^2 \left[\partial_U W_1\,(X^2 - Y^2) \ + \ 2\,\partial_U W_2\,XY\right] & \\ + \ 8\alpha' p^U \left[W_1(X\dot{X} - Y\dot{Y}) + W_2(X\dot{Y} + \dot{Y}X)\right] &= \ 0 \\ X'' - \ddot{X} \ + \ (2\alpha' p^U)^2\,[W_1 X - W_2 Y] &= \ 0 \\ Y'' - \ddot{Y} \ + \ (2\alpha' p^U)^2\,[W_2 X - W_1 Y] &= \ 0 \end{aligned} \tag{9}$$

and the constraints (3) take the form:

$$\pm\,\partial_\pm V_< = \frac{1}{\alpha' p^U}\left[(\partial_\pm X)^2 + (\partial_\pm Y)^2 + \sum_{i=3}^{D-2} (\partial_\pm X^i)^2\right] + \alpha' p^U \left[W_1\,(X^2 - Y^2) + 2\,W_2\,XY\right] \tag{10}$$

Let us analyze now the solutions of the string equations (9) and (10) for a closed string. The transverse coordinates obey the d'Alembert equation, with the solution

$$X^i(\sigma,\tau) = q^i + 2p^i\alpha'\tau + i\sqrt{\alpha'}\sum_{n\neq 0}\{\alpha_n^i \exp[in(\sigma-\tau)] + \tilde{\alpha}_n^i \exp[-in(\sigma+\tau)]\}/n, \qquad 3 \leq i \leq D-2. \tag{11}$$

For the X and Y components it is convenient to Fourier expand as

$$X(\sigma,\tau) = \sum_{n=-\infty}^{+\infty} \exp(in\sigma)\ X_n(\tau) \quad , \quad Y(\sigma,\tau) = \sum_{n=-\infty}^{+\infty} \exp(in\sigma)\ Y_n(\tau)$$

Then, eqs.(9) for X and Y yield

$$\begin{aligned} \ddot{X}_n + n^2 X_n - (2\alpha' p^U)^2\,[W_1 X_n - W_2 Y_n] &= \ 0 \\ \ddot{Y}_n + n^2 Y_n - (2\alpha' p^U)^2\,[W_2 X_n - W_1 Y_n] &= \ 0 \end{aligned} \tag{12}$$

where we consistently set $U = 2\alpha' p^U \tau$. Formally, these are two coupled one dimensional Schrödinger-like equations with τ playing the rôle of a spatial coordinate.

We study now the interaction of the string with the gravitational wave. For $2\alpha' p^U \tau < -T$, $W_{1,2}(\tau) = 0$ and therefore X, Y are given by the usual flat-space expansions

$$X(\sigma,\tau) = q^X_< + 2p^X_< \alpha'\tau + i\sqrt{\alpha'}\sum_{n\neq 0}\{\alpha^X_{n<}\exp[-in\tau)] - \tilde{\alpha}^X_{-n<}\exp[in\tau)]\}\exp[in\sigma]/n$$

$$Y(\sigma,\tau) = q^Y_< + 2p^Y_< \alpha'\tau + i\sqrt{\alpha'}\sum_{n\neq 0}\{\alpha^Y_{n<}\exp[-in\tau)] - \tilde{\alpha}^Y_{-n<}\exp[in\tau)]\}\exp[in\sigma]/n$$

These solutions define the initial conditions for the string propagation in $\tau \geq -\tau_0 \equiv -\frac{T}{2\alpha' p^U}$. In the language of the Schrödinger-like equations we have a two channel potential in the interval $-\tau_0 < \tau < +\tau_0$. We consider the propagation of the string when it approaches the singularity at $U = 0 = \tau$ from $\tau < 0$.

The general case when $W_1 \neq 0 \neq W_2$ is solved in [7]. Let us concentrate here on the case $W_2 \equiv 0$, $W_1(U) = \alpha\left[|U|^{-\beta} - |T|^{-\beta}\right]$ for $|U| < T$.

Eq.(12) can be approximated near $\tau = 0^-$ as

$$\ddot{X}_n - \frac{(2\alpha' p^U)^{2-\beta}}{|\tau|^\beta}\,\alpha\, X_n = 0$$

$$\ddot{Y}_n + \frac{(2\alpha' p^U)^{2-\beta}}{|\tau|^\beta}\,\alpha\, Y_n = 0$$

The behaviour of the solutions $X_n(\tau)$ and $Y_n(\tau)$ for $\tau \to 0$ depends crucially on the value range of β. Namely, i) $\beta > 2$, ii) $\beta = 2$, iii) $\beta < 2$.

When $\beta < 2$ the solution for $\tau \to 0^-$ behaves as

$$X(\sigma,\tau) \stackrel{\tau\to 0^-}{=} B^X(\sigma) + A^X(\sigma)\,\tau + O(|\tau|^{2-\beta})\,,$$

$$Y(\sigma,\tau) \stackrel{\tau\to 0^-}{=} B^Y(\sigma) + A^Y(\sigma)\,\tau + O(|\tau|^{2-\beta})\quad,\quad \beta \neq 1$$

[In the special case $\beta = 1$ one should add a term $0(\tau \ln|\tau|)$]. Here and in what follows, $B^X(\sigma), A^X(\sigma), B^Y(\sigma)$ and $A^Y(\sigma)$ are arbitrary functions depending on the initial data.

For $\beta < 2$, the string coordinates X, Y are always regular indicating that the string propagates smoothly through the gravitational-wave singularity U = 0. (Nevertheless, the velocities $\dot{X}$ and $\dot{Y}$ diverge at $\tau = 0$ when $1 \leq \beta < 2$).

For the case $\beta = 2$ the solution is [6]:

$$\begin{aligned} X(\sigma,\tau) &\stackrel{\tau\to 0^-}{=} B^X(\sigma)|\tau|^{(1-\sqrt{1+4\alpha})/2} + A^X(\sigma)|\tau|^{(1+\sqrt{1+4\alpha})/2} \\ Y(\sigma,\tau) &\stackrel{\tau\to 0^-}{=} B^Y(\sigma)|\tau|^{(1-\sqrt{1-4\alpha})/2} + A^Y(\sigma)|\tau|^{(1+\sqrt{1-4\alpha})/2} \end{aligned} \tag{13}$$

Let us now consider the case $\beta > 2$. We have [6]

$$X(\sigma,\tau) \quad \stackrel{\tau\to 0^-}{=} \quad B^X(\sigma)|\tau|^{\frac{\beta}{4}}\,\exp[K|\tau|^{1-\beta/2}] + A^X(\sigma)|\tau|^{\frac{\beta}{4}}\,\exp[-K|\tau|^{1-\beta/2}]\quad,$$

$$Y(\sigma,\tau) \overset{\tau\to 0^-}{=} A^Y(\sigma)|\tau|^{\frac{\beta}{4}} \cos\left[K|\tau|^{1-\beta/2} + C^Y(\sigma)\right] . \tag{14}$$

where

$$K = \frac{(2\alpha' p^U)^{1-\beta/2}}{\beta/2-1}\sqrt{\alpha} > 0 \quad ,$$

Let us now analyze the string behaviour near the singularity $\tau \to 0^-$ for $\beta \geq 2$. We see that for strong enough singularities ($\beta \leq 2$) the transverse coordinate X tends to infinity when the string approaches the singularity $\tau \to 0, U \to 0$. This means that the string does not cross the gravitational wave, since it does not reach the $U > 0$ region. The Y coordinate tends to zero oscillating, when $\tau \to 0$.

The string goes off to $X = \infty$, grazing the singularity plane $U = 0$ (therefore never crossing it). At the same time, the string oscillates in the Y direction, with an amplitude vanishing for $\tau \to 0^-$.

The spatial string coordinates $X^i(\sigma,\tau)$ $[3 \leq i \leq D-2]$ behave freely [eq.(11)]. The longitudinal coordinate $V(\sigma,\tau)$ follows from the constraint eqs.(10) and the solutions (13,14) for $X(\sigma,\tau), Y(\sigma,\tau)$ and $X^j(\sigma,\tau)$ $[3 \leq j \leq D-2]$. We see that for $\tau \to 0^-$, $V(\sigma,\tau)$ diverges as the square of the singular solutions (13,14).

Let us consider the spatial length element of the string, i.e. the length at fixed $U = 2\alpha' p^U \tau$, between two points (σ,τ) and $(\sigma + d\sigma, \tau)$,

$$ds^2 = dX^2 + dY^2 + \sum_{j=3}^{D-2}(dX^j)^2$$

For $\tau \to 0^-$ eqs.(13,14) yield

$$\begin{aligned} ds^2 &\overset{\tau\to 0^-}{=} \left[B^X(\sigma)'\right]^2 d\sigma^2 \; |\tau|^{1-\sqrt{1+4\alpha}} \quad \text{for } \beta = 2 , \\ ds^2 &\overset{\tau\to 0^-}{=} \left[B^X(\sigma)'\right]^2 d\sigma^2 \; |\tau|^{\beta/2} \exp\left[K|\tau|^{1-\beta/2}\right] \quad \text{for } \beta \geq 2. \end{aligned} \tag{15}$$

That is, the proper length between (σ_0,τ) and (σ_1,τ) is given by

$$\begin{aligned} \Delta s &\overset{\tau\to 0^-}{=} \left[B^X(\sigma_1) - B^X(\sigma_2)\right] \sqrt{|\tau|}^{1-\sqrt{1+4\tilde{\alpha}}} \to \infty \quad \text{for } \beta = 2 , \\ \Delta s &\overset{\tau\to 0^-}{=} \left[B^X(\sigma_1) - B^X(\sigma_2)\right] \; |\tau|^{\beta/4} \exp\left[K|\tau|^{1-\beta/2}\right] \to \infty \text{ for } \beta \geq 2. \end{aligned} \tag{16}$$

We see that $\Delta s \to \infty$ for $\tau \to 0^-$. That is, the string stretches infinitely when it approaches the singularity plane. This stretching of the string proper size also occurs for $\tau \to 0$ in the inflationary cosmological backgrounds as we shall see below.

Another consequence of eqs. (13,14) is that the string reaches infinity in a finite time τ. In particular, for σ-independent coefficients, eqs. (13,14) describe geodesic trajectories. The fact that for $\beta \geq 2$, a point particle (as well as a string) goes off to infinity in a finite τ indicates that the space-time is singular.

Finally, we would like to remark that the string evolution near the space-time singularity is a **collective motion** governed by the nature of the gravitational field. The (initial) state of the string fixes the overall σ-dependent coefficients $A^X(\sigma), B^X(\sigma)$, $A^Y(\sigma), B^Y(\sigma)$ [see eqs. (13,14)], whereas the τ-dependence is fully determined by the space-time geometry. In other words, the τ-dependence is the same for all modes n. In some directions, the string collective propagation turns to be an infinite motion (the escape direction X), whereas in the orthogonal direction (Y), the motion is oscillatory, but with a fixed (n-independent) frequency. In fact, these features are not restricted to singular gravitational waves, but **are generic** to strings in strong gravitational fields [see sec.(6) and refs.([8,1])].

For sufficiently weak spacetime singularities ($\beta_1 < 2$ and $\beta_2 < 2$), the string crosses the singularity and reaches the region $U > 0$. Therefore, outgoing scattering states and outgoing operators can be defined in the region $U > 0$. We explicitly found in [6,7] the transformation relating the ingoing and outgoing string mode operators. For the particles described by the quantum string states, this relation implies two types of effects as described in [11,1] for generic asymptotically flat spacetimes: (i) rotation of spin polarization in the (X,Y) plane, and (ii) **transmutation** between different particles. We computed in [6,7] the expectation values of the outgoing mass $M^2_>$ operator and of the mode-number operator $N_>$, in the ingoing ground state $|O_< >$. As for shockwaves (see [8]) , $M^2_>$ and $N_>$ have different expectation values than $M^2_<$ and $N_<$. This difference is due to the excitation of the string modes after crossing the space-time singularity. In other words, the string state is not an eigenstate of $M^2_>$, but an infinity superposition of one-particle states with different masses. This is a consequence of the particle transmutation which allows particle masses different from the initial one.

3. Strings and Multistrings in Cosmological Spacetimes and the Self-consistent string cosmology

Recently, several interesting progresses in the understanding of string propagation in cosmological spacetimes have been made [12−21]. The classical string equations of motion plus the string constraints were shown to be exactly integrable in D-dimensional de Sitter spacetime, and equivalent to a Toda-type model with a potential unbounded from below. In 2+1 dimensions, the string dynamics in de Sitter spacetime is exactly described by the sinh-Gordon equation.

Exact string solutions were systematically found by soliton methods using the linear system associated to the problem (the so-called dressing method in soliton theory) [13,14]. In addition, exact circular string solutions were found in terms of elliptic functions[15]. All these solutions describe one string, several strings or even an infinite number of different and independent strings. A single world-sheet simultaneously describes many different strings. This is a new feature appearing as a consequence of the interaction of the strings with the spacetime geometry. Here, interaction among the strings (like splitting and merging) is neglected, the only interaction is with the curved background. Different types of behaviour appear in the multistring solutions. For some of them the energy and proper size are bounded ('stable strings') while for many others the energy and size blow up for large radius of the universe ($R \to \infty$, 'unstable strings'). In addition, such stable and unstable string behaviours are exhibited by the ring solutions found in [17] for Friedmann-Robertson-Walker (FRW) universes and for power type inflationary backgrounds. In all these works, strings were considered as *test* objects propagating on the given *fixed* backgrounds.

We report here the recent results [5] further in the investigation of the physical properties of the string solutions above mentioned. We compute the energy-momentum tensor of these strings and we use it to find the back reaction effect on the spacetime. That is, we investigate whether these classical strings can sustain the corresponding cosmological background. This is achieved by considering **self-consistently**, the strings as matter sources for the Einstein (general relativity) equations (without the dilaton field), as well as for the string effective equations (beta functions) including the dilaton, the dilaton potential and the central charge term.

In spatially homogeneous and isotropic universes,

$$ds^2 = (dT)^2 - R(T)^2 \sum_{i=1}^{D-1} (dX^i)^2 \tag{17}$$

the string energy-momentum tensor $T_A^B(X)$, $(A, B = 1, \ldots D)$ for our string solutions takes the fluid form, allowing us to define the string pressure p through $-\delta_i^k \; p = T_i^k$ and the string energy density as $\rho = T_0^0$. The continuity equation $D^A \; T_A^B = 0$ takes then the form

$$\dot{\rho} + (D-1)\, H\, (p+\rho) = 0, \tag{18}$$

where $H \equiv \frac{d \log R}{dT}$. We consider $D = 1+1, D = 2+1$, and generic $D-$dimensional universes.

In 1+1 cosmological spacetimes we find the general solution of the string equations of motion and constraints for arbitrary expansion factor R . It consists of two families: one depends on two arbitrary functions $f_\pm(\sigma \pm \tau)$ and has *constant* energy density ρ and *negative* pressure $p = -\rho$. That is, a perfect fluid relation holds

$$p = (\gamma - 1)\rho \tag{19}$$

with $\gamma = 0$ in $D = 1+1$ dimensions. The other family of solutions depends on *two* arbitrary constants and describes a massless point particle (the string center of mass). This second solution has $p = \rho = u \; R^{-2} > 0$. This is a perfect fluid type relation with $\gamma = 2$. These behaviours fulfil the continuity equation (18) in $D = 2$.

In $2+1$ dimensions and for any factor R , we find that circular strings exhibit three different asymptotic behaviours :

- (i) **unstable** behaviour for $R \to \infty$ in inflationary universes (this corresponds to conformal time $\eta \sim \tau \to \tau_0$ with finite τ_0 and proper string size $S \sim R \to \infty$), for which the string energy $E_u \sim R \to \infty$ and the string pressure $p_u \simeq -E_u/2 \to -\infty$ is **negative** . This behavior dominates for $R \to \infty$ in inflationary universes.

- (ii) **Dual** to unstable behaviour for $R \to 0$. This corresponds to $\eta \sim (\tau - \tau_0)^{-1} \to +\infty$ for finite $\tau \to \tau_0$, $S \sim R \to 0$ (except for de Sitter spacetime where $S \to 1/H$), for which the string energy $E_d \sim 1/R \to \infty$ and the string pressure $p_d \simeq E/2 \to +\infty$, is **positive**.

- (iii) **Stable** for $R \to \infty$, (corresponding to $\eta \to \infty, \tau \to \infty, S =$ constant), for which the string energy is $E_s =$ constant and the string pressure vanishes $p_s = 0$.

Here the indices (u, d, s) stand for 'unstable', 'dual' and 'stable' respectively. The behaviours (i) and (ii) are related by the duality transformation $R \leftrightarrow 1/R$, the case (ii) being invariant under duality. In the three cases, we find perfect fluid relations (19) with the values of γ :

$$\gamma_u = 1/2 \quad , \quad \gamma_d = 3/2 \quad , \quad \gamma_s = 1 \; . \tag{20}$$

For a perfect gas of strings on a comoving volume R^2, the energy density ρ is proportional to E/R^2, which yields the scaling $\rho_u = u\, R^{-1}, \rho_d = d\, R^{-3}, \rho_s = s\, R^{-2}$. All densities and pressures obey the continuity equation (18) as it must be.

The 1+1 and 2+1 string solutions here described exist in any spacetime dimension. Embedded in D-dimensional universes, the $1+1$ and $2+1$ solutions describe straight strings and circular strings, respectively. In D-dimensional spacetime, strings may spread in $D-1$ spatial dimensions. Their treatment has been done asymptotically in [20]. We have three general asymptotic behaviours:

- (i) **unstable** for $R \to \infty$ in inflationary universes with $\rho_u = u\, R^{2-D}$, $p_u = -\rho_u/(D-1) < 0$

- (ii) **Dual** to unstable for $R \to 0$ with $\rho_d = d\, R^{-D}$, $p_d = \rho_d/(D-1) > 0$.

- (iii) **Stable** for $R \to \infty$, with $\rho_s = s\, R^{1-D}$, $p_s = 0$.

We find perfect fluid relations with the factors

$$\gamma_u = \frac{D-2}{D-1} \quad , \quad \gamma_d = \frac{D}{D-1} \quad , \quad \gamma_s = 1 \; . \tag{21}$$

This reproduces the two dimensional and three dimensional results for $D = 2$ and $D = 3$, respectively. The stable regime is absent for $D = 2$ due to the lack of string transverse modes there.

The dual strings behave as *radiation* (massless particles) and the stable strings are similar to *cold matter*. The unstable strings correspond to the critical case of the so called *coasting universe*. That is, classical strings provide a *concrete* realization of such cosmological models.

Strings continuously evolve from one type of behaviour to another, as is explicitly shown by our solutions [12,15]. For intermediate values of R, the string equation of state is clearly more complicated. We propose a formula of the type:

$$\rho = \left(u_R \, R + \frac{d}{R} + s \right) \frac{1}{R^{D-1}} \quad , \quad p = \frac{1}{D-1} \left(\frac{d}{R} - u_R \, R \right) \frac{1}{R^{D-1}} \tag{22}$$

where

$$\lim_{R \to \infty} u_R = \begin{cases} 0 & \text{FRW} \\ u_\infty \neq 0 & \text{Inflationary} \end{cases} \tag{23}$$

is qualitatively correct for all R and becomes exact for $R \to 0$ and $R \to \infty$. The parameter u_R varies smoothly with R and tends to the constant u_∞ for $R \to \infty$.

We stress here that we obtained the string equation of state from the exact string evolution in cosmological spacetimes.

Inserting the equation of state (22) in the Einstein-Friedmann equations of general relativity, we obtain a self-consistent solution for R as a monotonically increasing function of the cosmic time T

$$T = \sqrt{\frac{(D-1)(D-2)}{2}} \int_0^R dR \, \frac{R^{D/2-1}}{\sqrt{u_R \, R^2 + d + s \, R}} \tag{24}$$

where we set $R(0) = 0$. This string dominated universe starts at $T = 0$ with a radiation dominated regime $R(T) \overset{T \to 0}{\simeq} C_D \, (T)^{\frac{2}{D}}$, then the universe expands for large T as $R(T) \overset{T \to \infty}{\simeq} C'_D \, (T)^{\frac{2}{D-1}}$, as (cold) matter dominated universes. For example, at $D = 4$, R grows as $R \sim (T)^{\frac{2}{3}}$.

It must be noticed that an universe dominated by unstable strings (u_R) would yield $R(T) \overset{T \to \infty}{\simeq} C'_D \, (T)^{\frac{2}{D-2}}$, which is faster than (cold) matter dominated universes. However, this is not a self-consistent solution of the Einstein-Friedmann equations plus the string equations of motion, as shown in [5].

The unstable string solutions are called in this way since their energy and invariant length grow as R for large R. However, it must be clear that as *classical* string solutions they **never decay**.

Our self-consistent solution $R(T)$ yields the realistic matter behaviour $R(T) \overset{T \to \infty}{\sim} (T)^{\frac{2}{D-1}}$.

The *stable* strings (which behave as cold matter) are those dominating for $R \to \infty$. The 'dual' strings give $R(T) \overset{T \to 0}{\simeq} C_D \, (T)^{\frac{2}{D}}$, the radiation type behaviour. For intermediate R, the three types of string behaviours (unstable, dual and stable) are present. Their cosmological implications as well as those associated with string decay deserve investigation. For a thermodynamical gas of strings the temperature T as a function of R, scales as $1/R$ for small R (the usual radiation behaviour).

For the sake of completeness we analyze the effective string equations in [5]. These equations have been extensively treated in the literature and they are not our central aim.

It must be noticed that there is no satisfactory derivation of inflation in the context of the effective string equations. This does not mean that string theory is not compatible with inflation, but that the effective string action approach *is not enough* to describe inflation. The effective string equations are a low energy field theory approximation to string theory containing only the *massless* string modes. The vacuum energy scales to start inflation are typically of the order of the Planck mass where the effective string action approximation breaks down. One must also consider the *massive* string modes (which are absent from the effective string action) in order to properly get the cosmological condensate yielding inflation. De Sitter inflation does not emerge as a solution of the the effective string equations.

4. Conclusions

Strings in curved spacetimes show a rich variety of new behaviours unknown in flat spacetimes. The most spectacular effect is clearly given by the unstable strings with size and energy tending to infinity. This phenomenon appears both for singular plane wave spacetimes and for non-singular de Sitter spacetimes and for all other inflationary universes. We think that it is a generic feature for strings in strong gravitational fields.

Comparison of the energy-momentum behaviours in inflationary universes and singular plane-waves show interesting differences. We find for the fastest growing energy-momentum components in singular plane-waves ($\beta \geq 2$) [6]:

$$\begin{aligned} T^{VV}(\sigma,\tau) &\overset{\tau\to 0}{=} C_V\,[\xi(\tau)]^2 \int_0^{2\pi} [B^X(\sigma)]^2 \to \infty \\ T^{VX}(\sigma,\tau) &\overset{\tau\to 0}{=} C_X\,\xi(\tau) \int_0^{2\pi} B^X(\sigma) \to \infty \end{aligned} \tag{25}$$

where

$$\xi(\tau) \equiv |\tau|^{-\beta/4}\ \exp[K|\tau|^{1-\beta/2}] \to \infty$$

and C_V and C_X are constants.

A typical unstable string behaviour on an inflationary spacetime with scale factor $R(T) = a\ T^{\frac{k}{k+2}}$ $(k < 0)$ is as follows [2].

$$\begin{aligned} E &\overset{\tau\to 0}{=} \frac{C}{\alpha'}\ \tau^{k/2} = \frac{R}{\alpha'} \to +\infty \\ P &\overset{\tau\to 0}{=} -E/2 \to -\infty \\ S &\overset{\tau\to 0}{=} \tau^{k/2} \to +\infty \end{aligned} \tag{26}$$

where C is a constant and we considered a ring solution for simplicity.

The main difference is that in singular plane wave spacetimes only the null energy (conjugated to the null variable V) diverges. In the inflationary case, both the energy (T_0^0) and the pressure ($-T_i^i$, not summed) diverge and they blow up at the same rate. Obviously the kind of spacetime singularity is very different.

1. H.J. de Vega and N. Sánchez in "String Quantum Gravity and the Physics at the Planck Scale", Proceedings of the Erice Workshop held in June 1992. Edited by N. Sánchez, World Scientific, 1993. Pages 73-185, and references given therein.
2. H.J. de Vega and N. Sánchez in "Current Topics in Astrofundamental Physics", Proceedings of the Chalonge School, Erice 4-16 September 1994, Proceedings edited by N. Sánchez and A. Zichichi.
3. H. J. de Vega and N. Sánchez, Phys. Lett. **B 197**, 320 (1987).
4. H. J. de Vega and N. Sánchez, Nucl. Phys. **B 309**, 552 and 577 (1988).
5. H. J. de Vega and N. Sánchez, Phys. Rev. **D50**, 7202 (1994).
6. H. J. de Vega and N. Sánchez, Phys. Rev. **D 45** , 2783 (1992).
7. H. J. de Vega, M. Ramón Medrano and N. Sánchez, Class. and Quantum Grav. **10**, 2007 (1993).
8. H. J. de Vega and N. Sánchez, Nucl. Phys. **B 317**, 706 and 731 (1989). H. J. de Vega and N. Sánchez, Phys. Rev. Lett. (C), **65**, 1517 (1990).
9. G. Horowitz and A.R. Steif, Phys. Rev. Lett. **64**, 260 (1990) and Phys. Rev. **D 42** , 1950 (1990).
10. R. Güven, Phys. Lett. **B 191**, 275 (1987). D. Amati and K. Klimĉik, Phys. Lett. **B 210** , 92 (1988).
11. H. J. de Vega, M. Ramón Medrano and N. Sánchez, Nucl. Phys. **B 351**, 277 (1991), Nucl. Phys. **B 374**, 425 (1992) and Phys. Lett. **B 285**, 206 (1992).
12. H. J. de Vega and N. Sánchez, Phys. Rev. **D47**, 3394 (1993).
13. H. J. de Vega, A. V. Mikhailov and N. Sánchez, Teor. Mat. Fiz. **94** (1993) 232.
14. F. Combes, H. J. de Vega, A. V. Mikhailov and N. Sánchez, Phys. Rev. **D50**, 2754 (1994).
15. H. J. de Vega, A. L. Larsen and N. Sánchez, Nucl. Phys. **B 427**, 643 (1994).
16. I. Krichever, Funct. Anal. and Appl. **28**, 21 (1994), [Funkts. Anal. Prilozhen. **28**, 26 (1994)].
17. H. J. de Vega and I. L. Egusquiza, Phys. Rev. **D 49**, 763 (1994).
18. A. L. Larsen and N. Sánchez, Phys. Rev. **D 50**, 7493 (1994).
19. N. Sánchez and G. Veneziano, Nucl. Phys. **B 333**, 253 (1990).
20. M. Gasperini, N. Sánchez and G. Veneziano, Int. J. Mod. Phys. **A 6**, 3853 (1991) and Nucl. Phys. **B364**, 365 (1991).
21. H. J. de Vega and N. Sánchez, Int. J. Mod. Phys. **A 7**, 3043 (1992).