

DUALITY SYMMETRIES IN STRING THEORIES IN DIMENSION ≤ 4

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February 8, 2004

ABSTRACT

In this talk I review the status of various perturbative and non-perturbative duality symmetries in toroidally compactified heterotic string theories in dimensions four, three and two.

In this talk I shall be discussing (conjectured) duality symmetries in string theories in dimension ≤ 4 . The talk will be divided into three main parts. In the first part I shall discuss duality symmetries in four dimensional string theory. The second part of the talk will be on duality symmetries in three dimensional string theory. Finally, in the last part of the talk I shall discuss duality symmetries in two dimensional string theory. The talk will be based on refs.^{1,2,3}. Many related developments have been reported in other contributions to this volume.

Throughout this talk, a D dimensional string theory will refer to heterotic string theory compactified on a $(10 - D)$ dimensional torus. Such backgrounds are not phenomenologically interesting, since even for $D = 4$ it represents a theory with extended $N = 4$ supersymmetry, and as a result does not contain chiral fermions. However, the hope is that by studying symmetries of string theory around various backgrounds we may learn something about symmetries of a background independent formulation of string theory.

Also, such studies may give us a clue as to what to expect for a more realistic string compactification.

So let us begin our discussion with heterotic string theory compactified on a 6 dimensional torus.⁴ The massless scalar fields in this theory (also known as moduli fields) can be divided into two sets. One set parametrizes the coset $O(6, 22)/(O(6) \times O(22))$ and is labelled by a 28×28 matrix valued scalar field M satisfying the relations

$$M^T = M, \quad MLM^T = L, \quad (1)$$

where

$$L = \begin{pmatrix} \sigma_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \sigma_1 \\ & & & & -I_{16} \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}. \quad (2)$$

Physically M represents the internal components of the ten dimensional metric, anti-symmetric tensor field, and gauge fields belonging to the Cartan subalgebra of the gauge group $E_8 \times E_8$ or $SO(32)$. The second set of scalar fields is the axion-dilaton system, parametrizing the coset $SL(2, R)/SO(2)$ and labelled by a complex scalar field

$$\lambda \equiv \Psi + ie^{-\Phi} \equiv \lambda_1 + i\lambda_2, \quad (3)$$

where Ψ denotes the axion field obtained by dualizing the anti-symmetric tensor field $B_{\mu\nu}$ in four dimensions, and Φ denotes the usual dilaton field.

We shall now discuss the action of different duality symmetries on the scalar moduli fields. The action of the first kind of duality transformation, known as T -duality transformation, is as follows:

$$M \rightarrow \Omega M \Omega^T, \quad \lambda \rightarrow \lambda, \quad (4)$$

where $\Omega \in O(6, 22; Z)$. In practice this means that

$$\Omega L \Omega^T = L, \quad (5)$$

and that Ω preserves a 28 dimensional even, self-dual, Lorentzian lattice Λ_{28} with metric L . Although we can take for Λ_{28} any even, self-dual, Lorentzian

lattice, we shall work in a definite convention where the lattice is spanned by vectors of the form:

$$\begin{pmatrix} m_1 \\ \cdot \\ \cdot \\ \cdot \\ m_{12} \\ \vec{k} \end{pmatrix}, \quad m_i \in Z, \quad \vec{k} \in \Lambda_{E_8 \times E_8}. \quad (6)$$

Here $\Lambda_{E_8 \times E_8}$ denotes the $E_8 \times E_8$ root lattice. The important point to note from eq.(4) is that the T-duality transformations do not act on λ . From the definition (3) of λ it follows that

$$\langle \lambda \rangle = \frac{\theta}{2\pi} + \frac{i}{g^2}, \quad (7)$$

where θ denotes the usual vacuum angle, and g^2 denotes the string loop expansion parameter. Thus T -duality does not transform the string loop expansion parameter and is a symmetry that holds order by order in string perturbation theory. This makes T -duality easy to test.

The second kind of (conjectured) duality symmetry, known as the S -duality transformation, acts on M and λ as,

$$M \rightarrow M, \quad \lambda \rightarrow \frac{p\lambda + q}{r\lambda + s}, \quad p, q, r, s \in Z, \quad ps - qr = 1. \quad (8)$$

This constitutes an $SL(2, Z)$ group of transformations. Note, however, that this transformation does have non-trivial action on λ , and hence the string loop expansion parameter. We therefore cannot expect S -duality to hold order by order in string perturbation theory, and this makes this symmetry difficult to test.

In order to get a deeper insight into the role of these two duality transformations, we can examine their action on the gauge fields. The theory contains 28 abelian gauge fields $A_\mu^{(a)}$ ($1 \leq a \leq 28$, $0 \leq \mu \leq 3$) of which 16 arise from the Cartan subalgebra of the original $E_8 \times E_8$ or $SO(32)$ gauge fields, 6 arise from the components $G_{m\mu}$ ($4 \leq m \leq 9$) of the metric, and 6 arise from the components $B_{m\mu}$ of the anti-symmetric tensor field. Invariance

of the equations of motion of the low energy effective field theory forces us into the following transformation laws of the gauge fields. Under T -duality,

$$F_{\mu\nu}^{(a)} \rightarrow \Omega_{ab} F_{\mu\nu}^{(b)}, \quad (9)$$

where $F_{\mu\nu}^{(a)}$ denotes the field strength. On the other hand, under S -duality,

$$F_{\mu\nu}^{(a)} \rightarrow (r\lambda_1 + s)F_{\mu\nu}^{(a)} + r\lambda_2(ML)_{ab}\tilde{F}_{\mu\nu}^{(b)}, \quad (10)$$

where \tilde{F} denotes the dual field strength. From eq.(9) we see that T duality transformation does not mix electric fields with magnetic fields. This means that if we start with an elementary string excitation which carries electric charge but no magnetic charge, then the transformed state will also carry only electric charge. Thus the transformed state may be identified as another elementary string state. On the other hand, S -duality transformation does mix electric fields with magnetic fields. As a result if we start with an electrically charged elementary string state, then under S -duality transformation it goes into a state that carries both electric and magnetic charges in general. Such states must necessarily be identified as soliton states in the theory. Thus we see that S -duality transformation mixes elementary string states with solitons. This is another indication that S -duality transformation does not hold order by order in perturbation theory.

From this it would seem that it is almost impossible to design any test of S -duality, since we do not know how to calculate non-perturbative effects in string theory. Fortunately, due to the existence of an extended $N = 4$ supersymmetry in the theory under consideration, there are certain non-renormalization theorems which guarantee that for some of the quantities in this theory the tree level result is exact. This provides us with a laboratory for testing S -duality, since these quantities must be invariant under S -duality transformation. I shall not go into the details of these tests, but only mention here that conjectured S -duality in four dimensions have so far passed all such tests¹. Instead I shall now assume that S -duality is a valid symmetry of the four dimensional theory, and see what we can learn about lower dimensional theories starting from this assumption.

The general strategy that we shall use for this study is the following.

- We shall assume that S -duality is a symmetry of the four dimensional string theory, and that it does not get broken when we compactify one or more directions on a circle.

- We then combine the S -duality transformation of the original four dimensional theory with the known T -duality transformations of the lower dimensional theory, and find the minimal symmetry group that contains both. This can then be identified as the duality symmetry group of the lower dimensional theory.

We start our analysis for $D = 3$, *i.e.* heterotic string theory compactified on a seven dimensional torus². In this case the scalar moduli fields will be 30×30 matrix valued scalar fields \bar{M} satisfying the relations:

$$\bar{M}^T = \bar{M}, \quad \bar{M}^T \bar{L} \bar{M} = \bar{L}, \quad (11)$$

where \bar{L} is a 30×30 matrix of the form:

$$\bar{L} = \begin{pmatrix} \sigma_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \sigma_1 \\ & & & & -I_{16} \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}. \quad (12)$$

The independent components of the matrix valued scalar field \bar{M} are in one to one correspondence with the internal components of the metric, anti-symmetric tensor field, and gauge fields in the Cartan subalgebra of the $E_8 \times E_8$ gauge group. The other massless scalar field in this theory is the dilaton $\bar{\Phi}$. The theory also contains 30 massless $U(1)$ gauge fields $\bar{A}_{\bar{\mu}}^{(\bar{a})}$ ($1 \leq \bar{a} \leq 30$, $0 \leq \bar{\mu} \leq 2$), 16 of which come from the gauge fields in the Cartan subalgebra of $E_8 \times E_8$, 7 come from the components $G_{\bar{m}\bar{\mu}}$ of the ten dimensional metric ($3 \leq \bar{m} \leq 9$), and 7 come from the components $B_{\bar{m}\bar{\mu}}$ of the anti-symmetric tensor field. Since in three dimensions vector fields are dual to scalar fields, we can in fact trade in the 30 gauge fields for 30 scalar fields $\psi^{(\bar{a})}$.⁵ The set of scalar fields \bar{M} , $\bar{\Phi}$ and $\psi^{(\bar{a})}$ can now be combined into a 32×32 matrix valued scalar field \mathcal{M} , defined as,

$$\mathcal{M} = \begin{pmatrix} e^{-2\bar{\Phi}} + \psi^T \bar{L} \bar{M} \bar{L} \psi & -\frac{1}{2} e^{2\bar{\Phi}} \psi^T \bar{L} \psi & \psi^T \bar{L} \bar{M} \\ +\frac{1}{4} e^{2\bar{\Phi}} (\psi^T \bar{L} \psi)^2 & & +\frac{1}{2} e^{2\bar{\Phi}} \psi^T (\psi^T \bar{L} \psi) \\ -\frac{1}{2} e^{2\bar{\Phi}} \psi^T \bar{L} \psi & e^{2\bar{\Phi}} & -e^{2\bar{\Phi}} \psi^T \\ \bar{M} \bar{L} \psi & -e^{2\bar{\Phi}} \psi & \bar{M} + e^{2\bar{\Phi}} \psi \psi^T \\ +\frac{1}{2} e^{2\bar{\Phi}} \psi (\psi^T \bar{L} \psi) & & \end{pmatrix}. \quad (13)$$

\mathcal{M} satisfies

$$\mathcal{M}^T = \mathcal{M}, \quad \mathcal{M}^T \tilde{L} \mathcal{M} = \tilde{L}, \quad (14)$$

where \tilde{L} is a 32×32 matrix:

$$\tilde{L} = \begin{pmatrix} \sigma_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \sigma_1 \\ & & & & -I_{16} \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}. \quad (15)$$

One can easily check that \mathcal{M} has the same number of degrees of freedom as the set of scalar fields $\{\bar{M}, \bar{\Phi}, \psi^{(\bar{a})}\}$. In particular \mathcal{M} has 8×24 degrees of freedom, \bar{M} has 7×23 , $\bar{\Phi}$ has 1 and $\psi^{(\bar{a})}$ has 30 degrees of freedom.

Let us now investigate the duality symmetries of this theory. The target space duality transformation on the three dimensional fields take the form:

$$\bar{M} \rightarrow \bar{\Omega} \bar{M} \bar{\Omega}^T, \quad \psi^{(\bar{a})} \rightarrow \bar{\Omega}_{\bar{a}\bar{b}} \psi^{(\bar{b})}, \quad (16)$$

where $\bar{\Omega} \in O(7, 23; Z)$ is a 30×30 matrix satisfying the relations

$$\bar{\Omega} \bar{L} \bar{\Omega}^T = \bar{L}, \quad (17)$$

and that $\bar{\Omega}$ preserves a 30 dimensional even, self-dual, Lorentzian lattice Λ_{30} . For definiteness we shall take this lattice to be spanned by the vectors

$$\begin{pmatrix} m_1 \\ \cdot \\ \cdot \\ m_{14} \\ \vec{k} \end{pmatrix}, \quad m_i \in Z, \quad \vec{k} \in E_8 \times E_8. \quad (18)$$

The action of this transformation on the matrix \mathcal{M} is given by

$$\mathcal{M} \rightarrow \tilde{\Omega} \mathcal{M} \tilde{\Omega}^T, \quad (19)$$

where

$$\tilde{\Omega} = \begin{pmatrix} I_2 & \\ & \bar{\Omega} \end{pmatrix}. \quad (20)$$

Let us now turn to the S -duality transformations. Using the relationship between the fields \bar{M} , $\bar{\Phi}$ and $\psi^{(\bar{a})}$ of the three dimensional theory, and the

fields M , λ and $A_\mu^{(a)}$ of the four dimensional theory, one can study the effect of the S -duality transformation on the ‘three dimensional fields’. It can be shown that this transformation acts as,

$$\mathcal{M} \rightarrow \hat{\Omega} \mathcal{M} \hat{\Omega}^T, \quad (21)$$

where

$$\hat{\Omega} = \begin{pmatrix} p & 0 & 0 & -q \\ 0 & s & r & 0 \\ 0 & q & p & 0 \\ -r & 0 & 0 & s \\ & & & I_{28} \end{pmatrix}, \quad (22)$$

where p, q, r, s are the same integers appearing in eq.(8).

Examining eqs.(19)-(22) we see that the three dimensional T -duality transformations do not commute with the four dimensional S -duality transformations. Thus when we combine these two sets of transformations, we generate a much bigger symmetry group. In this case the symmetry group turns out to be $O(8, 24; Z)$. Under a general element of the group \mathcal{M} transforms as

$$\mathcal{M} \rightarrow \Omega \mathcal{M} \Omega^T, \quad (23)$$

where $\Omega \in O(8, 24; Z)$ is a 32×32 matrix, satisfying

$$\Omega \tilde{L} \Omega^T = \tilde{L}, \quad (24)$$

and that Ω preserves a 32 dimensional lattice Λ_{32} spanned by vectors of the form

$$\begin{pmatrix} m_1 \\ \cdot \\ \cdot \\ m_{16} \\ \vec{k} \end{pmatrix}, \quad m_i \in Z, \quad \vec{k} \in E_8 \times E_8. \quad (25)$$

Let us now turn to the study of duality symmetries in the two dimensional theory obtained by compactifying the heterotic string theory on an eight dimensional torus³. (For related work, see ref.⁶.) The moduli fields in this case are given by 32×32 matrices \tilde{M} satisfying

$$\tilde{M}^T = \tilde{M}, \quad \tilde{M}^T \tilde{L} \tilde{M} = \tilde{L}. \quad (26)$$

\widetilde{M} contains information about the internal components of the metric, anti-symmetric tensor field, and gauge fields. The T -duality group in this case is $O(8, 24; Z)$, and acts naturally on \widetilde{M} :

$$\widetilde{M} \rightarrow \widetilde{\Omega} \widetilde{M} \widetilde{\Omega}^T, \quad (27)$$

where $\widetilde{\Omega}$ is an $O(8, 24; Z)$ matrix defined above. S -duality transformations of the four dimensional theory, on the other hand, do not act naturally on \widetilde{M} , but on the variables dual to \widetilde{M} . As a result, there is a “conflict of interest” in trying to represent S - and T - duality transformations at the same time as local transformations on fields. To see an example of this, let us consider the two dimensional theory to be a result of compactifying the 2 and 3 directions of a four dimensional theory. Then the S -duality transformation acts naturally on the axion field obtained by dualizing the scalar field B_{23} in two dimensions. On the other hand, T -duality transformation acts naturally on the scalar field B_{23} itself.

Fortunately a solution to this problem has already been given in the literature⁷. Instead of working with the variable \widetilde{M} we work with a new variable $\widehat{\mathcal{V}}$ which is a $O(8, 24)$ valued function of the coordinate x and a real parameter v .^{*}We shall not give the construction of $\widehat{\mathcal{V}}$ in detail here; for details see³. Suffice it to say that $\widehat{\mathcal{V}}$ contains the same degrees of freedom as \widetilde{M} except for extra zero modes of dual potentials constructed from \widetilde{M} . Thus for a given $\widehat{\mathcal{V}}$ we can determine \widetilde{M} completely. The advantage of using the variable $\widehat{\mathcal{V}}$ instead on \widetilde{M} is that the action of both, the S and the T duality transformations are simple on $\widehat{\mathcal{V}}$. In particular, both the transformations can be represented as follows:

$$\widehat{\mathcal{V}}(x; v) \rightarrow g(v) \widehat{\mathcal{V}}(x; v) h(x; v), \quad (28)$$

where $g(v)$ is an $O(8, 24)$ valued function of the real variable v , and $h(x, v)$ is an $O(8, 24)$ valued function of x and v determined in terms of $g(v)$ and $\widehat{\mathcal{V}}(x, v)$. Thus all the duality transformations can be labelled by the elements $g(v)$. The T duality transformations correspond to constant $g(v)$:

$$g(v) = U, \quad U \in O(8, 24; Z). \quad (29)$$

^{*}The variable v corresponds to the spectral parameter that appears during the process of linearizing the equations of motion of \widetilde{M} .

On the other hand, S -duality transformations in the four dimensional theory corresponding to the non-compact directions 0 – 3 correspond to the choice

$$g(v) = \begin{pmatrix} p & 0 & 0 & -qv \\ 0 & s & rv^{-1} & 0 \\ 0 & qv & p & 0 \\ -rv^{-1} & 0 & 0 & s \\ & & & I_{28} \end{pmatrix}, \quad (30)$$

where p, q, r, s are integers satisfying $ps - qr = 1$.

Let us denote by $O(\widehat{8, 24})$ the infinite dimensional group represented by the set of all $O(8, 24)$ valued functions $g(v)$. Then, both the S - and the T - duality transformations correspond to appropriate subgroups of $O(\widehat{8, 24})$. The minimal duality group G of the two dimensional theory will be the subgroup of $O(\widehat{8, 24})$ generated by the S - and T - duality transformations. Some relevant subgroups of G are as follows.

- As already pointed out, the T -duality group of the two dimensional theory is generated by constant functions $g(v) = U$ where $U \in O(8, 24; \mathbb{Z})$.
- Consider the three dimensional theory obtained by compactifying the directions 3 – 9. Then according to our previous analysis, this theory has a duality group $O(8, 24; \mathbb{Z})$. This corresponds to the subgroup of G containing the elements $g(v) = VUV^{-1}$, where $U \in O(8, 24; \mathbb{Z})$, and

$$V = \begin{pmatrix} v & & \\ & v^{-1} & \\ & & I_{30} \end{pmatrix}. \quad (31)$$

In order to gain some further insight into the group G let us define $O(\widehat{8, 24}; \mathbb{Z})$ to be the discrete subgroup of $O(\widehat{8, 24})$ generated by $O(8, 24)$ valued functions $g(v)$ satisfying the constraint that it admits an expansion of the form:

$$g(v) = \sum_{n=-\infty}^{\infty} g_n v^{-n}, \quad (32)$$

with

$$g_n \Lambda_{32} \subset \Lambda_{32} \quad \forall n. \quad (33)$$

The matrices $g(v)$ representing the S - and T - duality transformations clearly satisfy this criteria. Thus we see that

$$G \subset O(8, \widehat{24}; Z) . \quad (34)$$

It is tempting to conjecture that G is the full group $O(8, \widehat{24}; Z)$. However, one can easily rule out this possibility by explicitly constructing examples of $O(8, \widehat{24}; Z)$ elements that are not in G . The examples of such elements are

$$g(v) = \begin{pmatrix} v & & \\ & v^{-1} & \\ & & I_{30} \end{pmatrix} . \quad (35)$$

The way to see that the above element is not in G is to note that all the generators of G satisfy the criteria that $g(v=1)g^{-1}(v=-1)$ is an element of $O(8, 24)$ that is continuously connected to the identity element. This is not true for the above element of $O(8, \widehat{24}; Z)$.

Even though $O(8, \widehat{24}; Z)$ elements of the form given in eq.(35) are not included in G , it is tempting to speculate that the full duality group in two dimension is $O(8, \widehat{24}; Z)$. This would imply that this duality group includes new elements like (35) that are not generated by known S - and T -duality transformations in higher dimensions.

Thus we see that heterotic string theory, compactified on a torus of dimension ≥ 6 , appears to have extra symmetries which are not visible in perturbation theory. This extra symmetry gets larger as we compactify more directions. The natural next step in this investigation would be to compactify one more dimension to get a one dimensional theory; or even to compactify two more dimensions by working in Euclidean space-time.

I wish to thank John H. Schwarz for many useful discussions, and for collaboration in part of the work reviewed in ¹. I also thank the organisers of the conference for hospitality during the extremely stimulating conference.

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