DUALITY SYMMETRIES AND PERTURBATIVE VECTORMULTIPLET COUPLINGS IN $N = 2$ HETEROTIC STRING VACUA*

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ABSTRACT

We study the low-energy effective Lagrangian of $N = 2$ heterotic string vacua at the classical and quantum level. The couplings of the vector multiplets are uniquely determined at the tree level, while the loop corrections are severely constrained by the exact discrete symmetries of the string vacuum. We evaluate the general transformation law of the perturbative prepotential and determine its form for the toroidal compactifications of six-dimensional $N = 1$ supersymmetric vacua.

1. Introduction

During the last months rigid $N = 2$ Yang-Mills gauge theories had been attracting a lot of interest; this developement was stimulated by Seiberg and Witten [1] who were able to solve non-perturbatively $N = 2$ supersymmetric Yang-Mills theory with a gauge group $SU(2)$ using the analytic properties of the $N = 2$ couplings in terms of the Higgs field. It is of interest to consider the analysis of Seiberg and Witten and its generalizations to larger gauge groups [2] in the context of Yang-Mills gauge theories coupled to $N = 2$ supergravity and in the context of $N = 2$ string theories. In this way one hopes to extract important informations about strong coupling phenomena in ($N = 2$) string theory and in particular about S-duality [3] which has been quite established for $N = 4$ heterotic string compactifications. In this talk we report on some work [4] about duality symmetries and the computation of perturbative couplings for $N = 2$ heterotic string theories. Related results along these lines were independently obtained in [5,6]. Very recently, also non-perturbative results were derived for $N = 2$ heterotic strings, based on a remarkable string-string duality between $N = 2$ heterotic versus type II strings [7] and also by discussing non-perturbative monodromies ala Seiberg and Witten [8].

Let us briefly recall how field dependend couplings arise in (supersymmetric) gauge theories [9]. Consider the spontaneous symmetry breakdown of a gauge group G (SU(2)) down to some subgroup H (U(1)) by a vacuum expectation value (vev) of a Higgs field a. At the one loop level one obtains the typical threshold behaviour

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$g_H^{-2}(a) = g_{\rm tree}^{-2} + \frac{b_H - b_G}{16\pi^2}\log a^2$, where the $b_{G,H}$ are the corresponding β-function coefficients; in the rigid case the tree level coupling $g_{\rm tree}^{-2}$ is usually field-independent. Clearly, the logarithmic singularity in the one-loop threshold function is due to states (gauge bosons) which become massless in the limit $a = 0$. In $N = 2$ Yang-Mills theories the gauge couplings are determined by an holomorphic prepotential $F(a)$: $g^{-2} = i\ {\rm Im}\Big(\frac{\partial^2 F(a)}{\partial a^2}\Big)$; the associated θ-angle reads $\theta = {\rm Re}\Big(\frac{\partial^2 F(a)}{\partial a^2}\Big)$. Now a denotes a $N = 2$ vector multiplet, respectively its complex scalar component. It is easy to integrate the above threshold formula deriving that $F(a)_{1-{\rm loop}} = a^2 + a^2(\log a^2 - {\rm const})$. The field a is not a gauge invariant quantity and the classical $\mathbf{Z}_2$ Weyl reflection, $a \to -a$ for $G = SU(2)$, induces the following symplectic transformation on the periods a and its dual $a_D = \frac{\partial F}{\partial a}$: $\begin{pmatrix} a \\ a_D \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a \\ a_D \end{pmatrix}$. This is the semiclassical monodromy corresponding to singularities due to massless gauge bosons at $a = 0$.

For $N = 2$ heterotic strings this scenario will be modified by gravitational and stringy effects. Besides the rigid (non-Abelian) gauge symmetries there will be two genuine $U(1)$ gauge groups associated to the graviphoton and to the supersymmetric spin one partner of the axion/dilaton field, denoted by S. The role of the Higgs fields, which break the non-Abelian gauge symmetries, is played by the moduli fields Φ^α of the $N = 2$ string vacua. The classical Weyl transformations on the Higgs fields are embedded into the perturbative target space duality symmetries acting on the moduli fields. According to the superconformal multiplet calculus of $N = 2$ supergravity the couplings of the $N = 2$ vector multiplets follow from an homogeneous prepotential $\mathcal{F}(S, \Phi^\alpha)$. $\mathcal{F}$ only gets perturbative contributions up to one loop plus non-perturbative contributions: $\mathcal{F}(S, \Phi^\alpha) = \mathcal{F}^{(0)}(S, \Phi^\alpha) + \mathcal{F}^{(1)}(\Phi^\alpha) + \mathcal{F}^{\rm NP}(S, \Phi^\alpha)$. The one-loop piece will be again multi-valued due to singularities of states which become massless at the points of enhanced gauge symmetries in the moduli space. Non-perturbatively, there will be singularities related to gauge configurations like monopoles and dyons; in addition one expects singularities due to non-pertutbative gravitational and/or stringy effects like massless black hole formation etc. at some points in the quantum moduli space. In this talk we entirely focus on the perturbative aspects of $N = 2$ string vacua.

To get a qualitative understanding about the form of the one-loop gauge couplings consider the simple case of one modulus T, where the target space duality group acting on T is given by $PSL(2, \mathbf{Z})$. The corresponding $U(1)$ gauge symmetry is enhanced to $SU(2)$ for $T = 1$. The Higgs field a is just the uniformazing variable $a = \frac{T-1}{T+1}$ [10,8] and the duality transformation $T \to 1/T$ acts as a Weyl reflection $a \to -a$ on the Higgs field. Thus we expect that in the vicinity of the critical point a the $U(1)$ gauge couplings behaves as $g^{-2} \sim \log a^2$. In order to get the proper transformation behaviour under the full target space duality group $PSL(2, \mathbf{Z})$, the gauge coupling will be given by an expression of the form $g^{-2} \sim \log(j(iT) - j(i))$ which reproduces $\log a^2$ for small a. We will explain how such kind of expression, which was obtained in [11] as a socalled free energy, is related to the second derivative of the one-loop holomorphic prepotential. Since $\mathcal{F}^{(1)}$ is multi-valued due to logarithmic singulairities, the target space duality transformation induces non-trivial one-loop monodromies.

2. Vector Couplings in $N = 2$ Supergravity

In $N = 2$ supersymmetric Yang-Mills theory the action is encoded in a holomorphic prepotential $F(X)$, where X^A $(A = 1, \ldots, n)$ denote the vector superfields and also the complex scalar components of such superfields. The local $N = 2$ supersymmetry requires an additional vector superfield X^0 in order to accomodate the graviphoton, but the scalar and the spinor components of this superfield do not lead to additional physical particles. Therefore, in the local case $F(X)$ is a holomorphic function of $n+1$ complex variables X^I $(I = 0, 1, \ldots, n)$, but it must be a *homogeneous* function of degree two [12]. According to the superconformal multiplet calculus, the physical scalar fields of this system parameterize an n-dimensional complex hypersurface. The embedding of this hypersurface can be described in terms of n complex coordinates z^A by letting X^I be proportional to some holomorphic sections $X^I(z)$ of the projective space. The resulting geometry for the space of physical scalar fields belonging to vector multiplets of an $N = 2$ supergravity is a *special* Kähler geometry [12,13], with a Kähler potential of the special form[a]

$$K(z, \bar{z}) = -\log\Big(i\bar{X}^I(\bar{z})\, F_I(X(z)) - iX^I(z)\, \bar{F}_I(\bar{X}(\bar{z}))\Big). \tag{1}$$

A convenient choice of inhomogeneous coordinates z^A are the *special* coordinates, defined b $z^A = X^A/X^0$, $A = 1, \ldots, n$, or, equivalently, $X^0(z) = 1$, $X^A(z) = z^A$. In this parameterization the Kähler potential can be written as

$$K(z, \bar{z}) = -\log\Big(2(\mathcal{F} + \bar{\mathcal{F}}) - (z^A - \bar{z}^A)(\mathcal{F}_A - \bar{\mathcal{F}}_A)\Big)\,, \tag{2}$$

where $\mathcal{F}(z) = i(X^0)^{-2}F(X)$.

The Lagrangian terms containing the kinetic energies of the gauge fields are

$$\mathcal{L}^{\rm gauge} \;=\; -\tfrac{i}{8}\left(\mathcal{N}_{IJ}\, F^{+I}_{\mu\nu} F^{+\mu\nu J} \;-\; \bar{\mathcal{N}}_{IJ}\, F^{-I}_{\mu\nu} F^{-\mu\nu J}\right)\,, \tag{3}$$

where $F^{\pm I}_{\mu\nu}$ denote the selfdual and anti-selfdual field-strength components and

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2i\frac{\mathrm{Im}(F_{IK})\,\mathrm{Im}(F_{JL})\, X^K X^L}{\mathrm{Im}(F_{KL})\, X^K X^L}\,. \tag{4}$$

Hence $\mathcal{N}$ is the field-dependent tensor that comprises the inverse gauge couplings $g^{-2}_{IJ} = \frac{i}{4}(\mathcal{N}_{IJ} - \bar{\mathcal{N}}_{IJ})$ and the generalized θ angles $\theta_{IJ} = 2\pi^2(\mathcal{N}_{IJ} + \bar{\mathcal{N}}_{IJ})$. Note the important identity $F_I = \mathcal{N}_{IJ}\, X^J$.

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[a]Here and henceforth we use the standard convention where $F_{IJ\ldots}$ denote multiple derivatives with respect to X of the holomorphic prepotential.

It is important to realize that, different functions $F(X)$ can lead to equivalent equations of motion. Such equivalence often involves the electric-magnetic duality of the field strengths rather than local transformations of the vector potentials A^I_μ. Following refs. [12,14,15], we define the tensors $G^{\pm}_{\mu\nu I}$ as $G^{+}_{\mu\nu I} = \mathcal{N}_{IJ} F^{+J}_{\mu\nu}$, $G^{-}_{\mu\nu I} = \bar{\mathcal{N}}_{IJ} F^{-J}_{\mu\nu}$. Then the set of Bianchi identities and equations of motion for the Abelian gauge fields can be written as $\partial^\mu(F^{+I}_{\mu\nu} - F^{-I}_{\mu\nu}) = 0$, $\partial^\mu(G^{+}_{\mu\nu I} - G^{-}_{\mu\nu I}) = 0$, which are invariant under the transformations

$$
\begin{aligned}
F^{+I}_{\mu\nu} &\longrightarrow \tilde{F}^{+I}_{\mu\nu} = U^I{}_J\, F^{+J}_{\mu\nu} + Z^{IJ}\, G^{+}_{\mu\nu J}\,. \\
G^{+}_{\mu\nu I} &\longrightarrow \tilde{G}^{+}_{\mu\nu I} = V_I{}^J\, G^{+}_{\mu\nu J} + W_{IJ}\, F^{+J}_{\mu\nu}\,,
\end{aligned}
\tag{5}
$$

where U, V, W and Z are constant, real, $(n+1)\times(n+1)$ matrices. However, the transformation eq.(5) must be a symplectic $Sp(2n+2,\mathbf{R})$ transformation, that is

$$
\begin{aligned}
&U^{\mathrm{T}}V - W^{\mathrm{T}}Z = V^{\mathrm{T}}U - Z^{\mathrm{T}}W = \mathbf{1}\,, \\
&U^{\mathrm{T}}W = W^{\mathrm{T}}U\,, \qquad Z^{\mathrm{T}}V = V^{\mathrm{T}}Z\,.
\end{aligned}
\tag{6}
$$

Next, consider the transformation rules for the scalar fields. $N=2$ supersymmetry relates the X^I to the field strengths $F^{+I}_{\mu\nu}$, while the F_I are related to the $G^{+\mu\nu}_I$. The $Sp(2n+2,\mathbf{R})$ act on the scalar fields as

$$
\begin{aligned}
\tilde{X}^I &= U^I{}_J\, X^J + Z^{IJ}\, F_J, \\
\tilde{F}_I &= V_I{}^J\, F_J + W_{IJ}\, X^J,
\end{aligned}
\tag{7}
$$

Owing to the symplectic conditions eqs.(6), the quantities $\tilde{F}_I$ can be written as the derivative of a new function $\tilde{F}(\tilde{X})$ with respect to the new coordinate $\tilde{X}^I$. Checking whether a particular symplectic transformations corresponds to a symmetry transformation on has to demand that substituting $\tilde{X}$ for X in $F_I(X)$ induces precisely the symplectic transformation specified in the second formula of eq.(7).

The transformation rule for the tensor $\mathcal{N}$ is precisely as in the rigid case, namely $\tilde{\mathcal{N}}_{IJ} = (V_I{}^K \mathcal{N}_{KL} + W_{IL})[(U + Z\mathcal{N})^{-1}]^L{}_J$. Three particular subgroups of the $Sp(2n+2,\mathbf{R})$ will be relevant to our following discussion The first subgroup contains the classical target-space duality transformations which are symmetries of the tree-level Lagrangian. One can easily see that the Lagrangian is left invariant by the subgroup that satisfies $W = Z = 0$ and $V^T = U^{-1}$. For the second subgroup, we continue to demand $Z = 0$ but relax the $W = 0$ condition; according to eq. (6), we then should have $V^{\mathrm{T}} = U^{-1}$ and $W^{\mathrm{T}}U$ should be a symmetric matrix. These conditions lead to *semiclassical* transformations of the form

$$
\begin{aligned}
\tilde{X}^I &= U^I{}_J\, X^J\,, \qquad & \tilde{F}_I &= [U^{-1}]^J{}_I\, F_J + W_{IJ}X^J\,, \\
\tilde{F}^{\pm I}_{\mu\nu} &= U^I{}_J\, F^{\pm J}_{\mu\nu}\,, \qquad & \tilde{\mathcal{N}} &= [U^{-1}]^{\mathrm{T}}\mathcal{N}U^{-1} + WU^{-1}\,,
\end{aligned}
\tag{8}
$$

which can always be implemented as Lagrangian symmetries of the vector fields A^I_μ. The last term in the last equation in (8) amounts to a constant shift of the theta angles; at the quantum level, such shifts are quantized and hence the symplectic group must be restricted to $Sp(2n+2,\mathbf{Z})$. We will see that such shifts in the θ-angle do occur whenever the one-loop gauge couplings have logarithmic singularities at special points in the moduli space where massive modes become massless. Therefore, these symmetries are related to the semi-classical (one-loop) monodromies around such singular points. The third subgroup contains elements that interchange the field-strength tensors $F^I_{\mu\nu}$ and $G_{\mu\nu I}$ and correspond to electric-magnetic dualities. These transformations are defined by $U = V = 0$ and $W^{\mathrm{T}} = -Z^{-1}$, which yields $\tilde{\mathcal{N}} = -W\,\mathcal{N}^{-1}\,W^{\mathrm{T}}$, so that they give rise to an inversion of the gauge couplings and hence must be non-perturbative in nature. In the heterotic string context, such transformations are often called S-dualities because of the way they act upon the dilaton field S.

3. $N=2$ Heterotic Strings

3.1. Spectrum in case of toroidal compactification

For a heterotic string vacuum with $N=2$ space-time supersymmetry the algebra implies that the internal right-moving $c=9$ SCFT contains a free complex $N=1$ superfield whose bosonic components we denote by $\partial X^{\pm}(z)$. The massless spectrum of a heterotic $N=2$ vacuum always comprises the graviton ($G_{\mu\nu}$), the antisymmetric tensor ($B_{\mu\nu}$) and the dilaton (D), created by vertex operators of the form $\bar\partial X_\mu(\bar z)\,\partial X_\nu(z)$ (at zero momentum), and two gravitini and two dilatini, created by vertex operators $\bar\partial X_\mu(\bar z)\,V^i_\alpha(z)$. In addition, there are always two Abelian gauge bosons $A^{\pm}_\mu$ with vertex operators $\bar\partial X_\mu(\bar z)\,\partial X^{\pm}(z)$, which generate the gauge group $[U(1)_R]^2$ (the suffix R indicates that these groups originate from the dimension-one operators $\partial X^{\pm}(z)$ of the right-moving sector). One linear combination is the graviphoton, which is the spin-1 gauge boson of the $N=2$ supergravity multiplet that also contains the graviton and two gravitini. The dilaton together with the antisymmetric tensor $B_{\mu\nu}$, the two dilatini and the remaining $U(1)$ vector are naturally described by an $N=2$ *vector-tensor* multiplet [16,4]. In a dual description, where the antisymmetric tensor is replaced by a pseudo-scalar (axion) a, the degrees of freedom form a $N=2$ vector supermultiplet where the dilaton and the axion combine into a complex scalar $S = e^D + ia$.

Apart from the two Abelian gauge bosons we just discussed, the massless spectrum of heterotic string vacua contain further gauge bosons A^a_μ, which are always members of $N=2$ vector multiplets. Their superpartners are two gaugini $\lambda^a_{i\,\alpha}$ and a complex scalar C^a and the vertex operators for a generic vector multiplet are given by

$$\Big(A^a_\mu, \lambda^a_{i\,\alpha}, C^a\Big) \sim \Big(J^a(\bar z)\,\partial X_\mu(z), J^a(\bar z)\,V_{i\,\alpha}(z), J^a(\bar z)\,\partial X^{\pm}(z)\Big)\,, \tag{9}$$

where $J^a(\bar z)$ are dimension $(1,0)$ operators that together comprise a left-moving Kač-Moody current algebra. Their zero modes generate a non-Abelian gauge group G.

The scalar fields in the Cartan subalgebras of non-Abelian factors $G_{(a)} \subset G$ as well as the scalars of any Abelian factor in G correspond to flat directions of the $N = 2$ scalar potential. Their vertex operators are truly marginal operators of the SCFT and the corresponding space-time vacuum expectation values are free parameters which continuously connect a family of string vacua.

¿From now on we focus on the particular subclass of four-dimensional $N = 2$ heterotic vacua, namely compactifications of six-dimensional $N = 1$ heterotic vacua on a two-torus T^2. The right-moving coordinates of the torus are given by the operators $\partial X^{\pm}(z)$ discussed previously, but now there also exist two free complex left-moving operators $\bar{\partial} X^{\pm}(\bar{z})$, which can be used to build vertex operators for the two complex moduli of the torus $\bar{\partial} X^{\pm}(\bar{z})\, \partial X^{\pm}(z)$. The moduli of T^2 are commonly denoted by $T = 2(\sqrt{G} + iB)$ and $U = (\sqrt{G} - iG_{12})/G_{11}$, where G_{ij} is the metric of T^2, $\sqrt{G}$ its determinant and B the constant antisymmetric-tensor background; U describes the deformations of the complex structure while T parameterizes the deformations of the area and the antisymmetric tensor, respectively. The moduli space spanned by T and U is determined by the Narain lattice of T^2 [17]:

$$\mathcal{M}_{T,U} = \left(\frac{SO(2,2)}{SO(2)\times SO(2)}\right)_{T,U} \simeq \left(\frac{SU(1,1)}{SU(1)}\right)_T \otimes \left(\frac{SU(1,1)}{SU(1)}\right)_U . \tag{10}$$

All physical properties of the two-torus compactifications are invariant under the group $SO(2,2,\mathbf{Z})$ of discrete duality transformations [18], which comprise the $T \leftrightarrow U$ exchange and the $PSL(2,\mathbf{Z})_T \times PSL(2,\mathbf{Z})_U$ dualities, which acts on T and U as

$$T \rightarrow \frac{aT - ib}{icT + d}\,, \qquad U \rightarrow \frac{a'U - ib'}{ic'U + d'}\,, \tag{11}$$

where the parameters $a, \ldots, d'$ are integers and constrained by $ad - bc = a'd' - b'c' = 1$.

T and U are the spin-zero components of two additional $U(1)$ $N = 2$ vector supermultiplets. The necessary enlargement of the Abelian gauge symmetry is furnished by vertex operators of the form $\bar{\partial} X^{\pm}(\bar{z})\, \partial X_\mu(z)$ which generate the gauge group $[U(1)_L]^2$. At special points in the (T, U) moduli space, additional vector fields become massless and the $U(1)_L^2$ becomes enlarged to a non-Abelian gauge symmetry. In particular, along the critical $T = U$ line, there are two additional massless gauge fields and the $U(1)_L^2$ becomes $[SU(2) \times U(1)]_L$. Similar critical lines exist for $T \equiv U \pmod{SL(2,\mathbf{Z})}$, *i.e.*, $T = (aU - ib)/(icU + d)$ for some integer a, b, c, d with $ad - bc = 1$. When two such lines intersect, each line brings with it a pair of massless gauge fields and the gauge symmetry becomes enhanced even further; the enhanced group may be determined by simply counting the intersecting critical lines [11]. For example, the point $T = U = 1$ lies at the intersection of two critical lines, namely $T = U$ and $T = 1/U$, and hence has four extra gauge bosons. The corresponding gauge symmetry is $SU(2)_L^2$. On the other hand, three critical lines $T = U$, $T = 1/(U - i)$ and $T = (iU + 1)/U$ intersect at the critical point $T = U = \rho = e^{2\pi i/12}$, where one therefore has six massless gauge bosons in addition to the $U(1)_L^2$; this enhances the gauge symmetry all the way to an $SU(3)_L$.

3.2. Classical vector couplings

The couplings of the dilaton vector multiplet are independent of the properties of the internal SCFT and thus universal at the string tree level; in particular, the dilaton does not mix with any of the other scalar fields in the spectrum of the EQFT. Furthermore, the axion is subject to a continuous Peccei-Quinn symmetry, which implies that the Kähler potential is only a function of $(S+\bar{S})$. Both properties together imply that the moduli space contains the dilaton field S as the complex coordinate of a separate $SU(1,1)/U(1)$ factor. The only special Kähler manifold of any dimension $n>1$ that satisfies this constraint is the symmetric space [19]

$$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n-1)}{SO(2)\times SO(n-1)}\,, \tag{12}$$

with a prepotential (up to symplectic reparametrizations)

$$F(X) = -\frac{X^1X^2X^3}{X^0}\,. \tag{13}$$

The moduli Φ of the previous section are identified with $S=-i\frac{X^1}{X^0}$, $T=-i\frac{X^2}{X^0}$, $U=-i\frac{X^3}{X^0}$. The gauge couplings $\mathcal{N}_{IJ}$ for for the vector superpartners of the moduli scalars can be derived in a straightforward way and are given in ref.[4]. One observes that all these gauge couplings are non-holomorphic functions of the moduli, which is a direct consequence of the mixing between the graviphoton and the Abelian vector superpartners of the moduli scalars. Furthermore, one can easily see that most of the gauge couplings are proportional to the dilaton's expectation value and hence the corresponding gauge couplings become weak in the large-dilaton limit. The exceptions are $\mathcal{N}_{SS}$, which is proportional to $S+\bar{S}$ and the off-diagonal matrix elements $\mathcal{N}_{S\,(T,U)}$, which are of the order $O(1)$ in the large-dilaton limit. On the other hand, from the string theory we know that *all* the *physical* low-energy couplings become weak in the large-dilaton limit, which suggests that the strongly-coupled $F^{+S}_{\mu\nu}$ field strength in the dilaton $N=2$ superfield should be replaced with its dual (which is weakly coupled in the large-dilaton limit). In $N=2$ terms, this is achieved by the symplectic transformation $(X^I,F_J)\to(\hat{X}^I,\hat{F}_J)$ where $\hat{X}^I=X^I$ for $I\neq 1$, $\hat{X}^1=F_1$, $\hat{F}_I=F_I$ for $I\neq 1$, $\hat{F}_1=-X^1$. The new coordinates $\hat{X}^I$ are, however, not independent, as they no longer depend on X^1. This reflects itself in the constraint $\eta_{IJ}\hat{X}^I\hat{X}^J \stackrel{\text{def}}{=} \hat{X}^1\hat{X}^0 + \hat{X}^2\hat{X}^3 - \hat{X}^i\hat{X}^i - \hat{X}^a\hat{X}^a = 0$ (the first equality here defines the symmetric matrix η), which can be easily verified by an explicit calculation. Consequently the matrix $\mathcal{S}^I{}_J(X) = \partial\hat{X}^I/\partial X^J$ has zero determinant and hence no meaningful prepotential $\hat{F}(\hat{X})$ can be defined [5]. Nevertheless, the gauge couplings can be computed in the new basis. One finds

$$\hat{\mathcal{N}}_{IJ} = -2i\bar{S}\,\eta_{IJ} + 2i(S+\bar{S})\,\frac{\eta_{IK}\,\eta_{JL}(\hat{z}^K\hat{\bar{z}}^L + \hat{\bar{z}}^K\hat{z}^L)}{\hat{z}^K\eta_{KL}\hat{\bar{z}}^L}\ , \tag{14}$$

which has a rather symmetric form in terms of special coordinates $\hat{z}^P \equiv \hat{X}^P/\hat{X}^0$. In particular, in the new basis, all the Im $\hat{\mathcal{N}}_{IJ}$ are proportional to $S+\bar{S}$ and hence *all* the gauge couplings become weak in the large-dilaton limit.

The basis $(\hat{X}^I, \hat{F}_J)$ is particularly well suited for the treatment of the the target-space-duality symmetries of generic $N=2$ heterotic string vacua since the classical Lagrangian is manifestly invariant under symplectic transformations with $\hat{W} = \hat{Z} = 0$ and $\hat{U}$ (and thus $\hat{V}$) belonging to $SO(2,2)$. Under this symmetry, the periods thus transform according to $\hat{X}^I \to \hat{U}^I{}_J\,\hat{X}^J$, $\hat{F}_I \to [\hat{U}^{-1}]^J{}_I\,\hat{F}_J$, while the field strengths and vector potentials also transform according to the $\hat{U}$ matrix. The dilaton field remains invariant at the classical level. Specifically, the corresponding symplectic matrices (in the basis $(\hat{X}^I, \hat{F}_I)$), for $PSL(2,\mathbf{Z})_T \times PSL(2,\mathbf{Z})_U$ are given by

$$\hat{U} = \begin{pmatrix} d & 0 & c & 0 \\ 0 & a & 0 & -b \\ b & 0 & a & 0 \\ 0 & -c & 0 & d \end{pmatrix}\ , \quad \hat{V} = (\hat{U}^T)^{-1} = \begin{pmatrix} a & 0 & -b & 0 \\ 0 & d & 0 & c \\ -c & 0 & d & 0 \\ 0 & b & 0 & a \end{pmatrix}\ , \tag{15}$$

while $\hat{W} = \hat{Z} = 0$.

3.3. 1-loop vector couplings

Since the dilaton serves as the loop-counting parameter of the heterotic string the one-loop prepotential cannot depend on the S-field. For the same reason, any possible two-loop or higher-loop corrections would have to be proportional to negative powers of the dilaton and because of the continous Peccei-Quinn symmetry (which persists to all orders in the perturbation theory), such corrections would have to involve the negative powers of the $(S+\bar{S})$ combination rather than just S. On the other hand, $\bar{S}$ clearly cannot appear in the holomorphic prepotential $\mathcal{F}(\Phi)$ and hence in string theory, all perturbative corrections to the prepotential stop at the one-loop level, in full analogy to the field-theoretical expansion, which also terminates at the one-loop order. The one-loop prepotential cannot be arbitrary functions of the moduli, since they should respect any exact duality symmetry a string vacuum might have. In the previous section we saw that the tree-level geometry of the moduli space is invariant under the $SO(2,2)$ isometry group, and from string theory we know that transformations belonging to a discrete $SO(2,2,\mathbf{Z})$ subgroup of this isometry group are in fact exact symmetries of string vacua to all orders in perturbation theory. The goal of this and the following sections is to find the precise conditions such exact symmetries impose on the one-loop prepotential.

We write the holomorphic prepotential for the homogeneous variables X^I as

$$F(X) = H^{(0)}(X) + H^{(1)}(X)\,, \tag{16}$$

where $H^{(0)}(X)$ is the tree-level prepotential (13) while $H^{(1)}(X) = -i(X^0)^2\, h^{(1)}$ represents the one-loop contribution. Both functions are homogeneous of second degree and $H^{(1)}$ does not depend on X^1. However, the most convenient variables for our purpose are again $\hat{X}^I$ and $\hat{F}_I$. (Since the one-loop prepotential $H^{(1)}$ does not depend on X^1, it follows that $\hat{X}^1 = F_1$ is not modified by loop corrections.) In terms of the $(\hat{X}^I, \hat{F}_I)$ variables, this means that the $\hat{X}^I$ should transform exactly as in the classical theory, without any perturbative corrections. On the other hand, the corresponding transformation rules for the $\hat{F}_I$ become modified at the one-loop level since the Lagrangian is no longer invariant. Instead the transformation rules have to generate discrete shifts in various θ angles due to monodromies around semi-classical singularities in the moduli space where massive string modes become massless. We have anticipated this situation in eqs. (8): Instead of the classical transformation rules, in the quantum theory, $(\hat{X}^I, \hat{F}_I)$ transform according to

$$\hat{X}^I \;\rightarrow\; \hat{U}^I{}_J\,\hat{X}^J, \qquad \hat{F}_I \;\rightarrow\; \hat{V}_I{}^J\,\hat{F}_J \;+\; \hat{W}_{IJ}\,\hat{X}^J\;, \tag{17}$$

where $\hat{V} = (\hat{U}^{\mathrm{T}})^{-1}$, $\hat{W} = \hat{V}\Lambda$, $\Lambda = \Lambda^{\mathrm{T}}$ and $\hat{U}$ belongs to $SO(2,2)$. Classically, $\Lambda = 0$, but in the quantum theory, Λ is an arbitrary real symmetric matrix, which should be integer valued so that the ambiguities in the θ angles are discrete ($\delta\theta = \hat{W}\hat{U}^{-1} = \hat{V}\Lambda\hat{V}^{\mathrm{T}}$). In particular, for a closed monodromy around a singularity, $\hat{X}^I \rightarrow \hat{X}^I$ *i. e.* $\hat{U} = 1$, but $\Lambda \neq 0$ and $\hat{F}_I \rightarrow \hat{F}_I + \Lambda_{IJ}\hat{X}^J$. We recall that the prepotential itself is in general not invariant under a symmetry of the equations of motion corresponding to the effective action, as one can easily verify for the tree-level results of the previous section, but the period transformation rules are correctly induced by the transformations of the coordinates. Therefore one obtains the corresponding transformation rule for $H^{(1)}$,

$$H^{(1)}(\tilde{X}) \;= H^{(1)}(X) \;+\; \tfrac{1}{2}\Lambda_{IJ}\hat{X}^I\hat{X}^J. \tag{18}$$

Note that the dilaton field does not appear anywhere in this formula. To put the symmetry relation (18) in its proper context, it is important to keep in mind that $H^{(1)}$ should have a logarithmic singularity whenever an otherwise massive string mode becomes massless; therefore, as an analytic function of (X^0, X^2, X^3), $H^{(1)}(X)$ is generally multi-valued. According to eq. (18), the ambiguities of $H^{(1)}$ amount to quadratic polynomials in the variables $\hat{X}^I$ with some discrete real coefficients; indeed, under a closed monodromy one generally has $H^{(1)} \rightarrow H^{(1)} + \frac{1}{2}\Lambda_{IJ}\hat{X}^I\hat{X}^J$ even though the fields (X^0, X^2, X^3) remain unchanged. However, modulo these ambiguities, $H^{(1)}$ should be invariant under all the exact symmetries of the perturbative string theory. This is the main result of this section.

Let us now turn our attention to the dilaton field S. In perturbative string theory, the dilaton vertex and its superpartners have fixed relations to the vector-tensor multiplet. However, the duality relation between this vector-tensor multiplet and the Abelian vector multiplet containing $S = -iX^1/X^0$ is not fixed but suffers from perturbative corrections in both string theory and field theory. Therefore, while the vector-tensor multiplet is inert under all the perturbative symmetries of the string's vacuum, the S field is only invariant classically but has non-trivial transformation properties at the one-loop level of the quantum theory. Indeed, using the relation $X^1 = -\hat{F}_1 = -iS\hat{X}^0$, it is easy to show that the transformation rules (17) imply

$$S \;\rightarrow\; \tilde{S} \;=\; S \;+\; \frac{i\hat{V}_1{}^J \left(H_J^{(1)} + \Lambda_{JK}\hat{X}^K\right)}{\hat{U}^0{}_I\,\hat{X}^I}\,, \tag{19}$$

which in turn is sufficient to assure the correct transformation properties of all the $\hat{F}_I$ and not just the $\hat{F}_1$.

The transformation formula for $H^{(1)}(X)$ can be also translated into the aquivalent transformation rules for $h^{(1)}$. For the case at hand, the target-space duality group is $SO(2,2,\mathbf{Z})$ consisting of the $T \leftrightarrow U$ exchange and of the $PSL(2,\mathbf{Z})_T$ and $PSL(2,\mathbf{Z})_U$ dualities whose action is described by eq. (15). Substituting these dualities into the general transformation law (18), we find

$$T \;\rightarrow\; \frac{aT-ib}{icT+d}, \qquad h^{(1)}(T,U) \rightarrow \frac{h^{(1)}(T,U) + \Xi(T,U)}{(icT+d)^2}, \tag{20}$$

and a similar set of transformations (with T and U interchanged) for the $PSL(2,\mathbf{Z})_U$. The appearance of $\Xi = \frac{i}{2}\Lambda_{IJ}\hat{X}^I\hat{X}^J/(\hat{X}^0)^2$ in these formulæ complicates the symmetry properties of the one-loop moduli prepotential, which would otherwise be a modular function of weight -2 with respect to both T and U dualities. However, Ξ is a quadratic polynomial in the variables $(1, iT, iU, TU)$ and hence $\partial_T^3\Xi = \partial_U^3\Xi = 0$; also, it is a mathematical fact that the third derivative of a modular function of weight -2 is itself a modular function of weight $+4$ even though the derivative is ordinary rather than covariant. From these two observations, we immediately learn that $\partial_T^3 h^{(1)}(T,U)$ is a single-valued modular function of weight $+4$ under the T-duality and of weight -2 under the U-duality and there are no anomalies in its modular transformation properties; the same is of course true for the $\partial_U^3 h^{(1)}$, with the two modular weights interchanged.

The exact analytic form of a modular function can often be completely determined from the knowledge of its singularities and its asymptotic behavior when $T \rightarrow \infty$ or $U \rightarrow \infty$. It was argued in ref. [20] that the gauge couplings of an $N = 1$ orbifold cannot grow faster than a power of T or U in any decompactification limit and the same argument applies here to the one-loop prepotential $h^{(1)}$ and any of its derivatives. Let us therefore consider the singularity structure of the $h^{(1)}(T,U)$.

The gauge couplings of the $[U(1)_L]^2$ containing the vector partners of T and U become singular whenever there are additional massless particles charged under this group. As discussed in section 2, this happens along the complex lines $T \equiv U$, where the $U(1)^2_L$ group is enlarged to an $SU(2)\times U(1)$; when such lines intersect each other, the group is further enlarged to an $SU(2)\times SU(2)$ (at $T\equiv U\equiv 1$) or an $SU(3)$ (at $T\equiv U\equiv\rho=e^{2\pi i/12}$). However, for a fixed generic value of U, the only singularities in the complex T-plane (or rather half-plane $\mathrm{Re}\,T>0$) are at $T\equiv U$ while the points $T\equiv 1\not\equiv U$ and $T\equiv\rho\not\equiv U$ are perfectly regular; the same is of course true for the singularities in the U-plane (or rather half-plane) when T is held fixed at a generic value. Hence, for generic T or U but small $T-U$,

$$h(T\approx U) \;=\; \tfrac{1}{16\pi^2}\,(T-U)^2\log(T-U)^2 \;+\; \text{regular}, \tag{21}$$

although the "regular" term here is only regular when $T\approx U\not\equiv 1,\rho$. Note that h is singular but finite when $T\approx U$; its third derivatives $\partial^3_T h=\partial^3_T h^{(1)}$ and $\partial^3_U h=\partial^3_U h^{(1)}$ have simple poles at that point and similar poles whenever $T\equiv U \pmod{SL(2,\mathbf{Z})}$. This fact, plus all the other properties of the functions $\partial^3_{T,U}h^{(1)}(T,U)$ we have stated above, allow us to uniquely determine

$$\begin{aligned}
\partial^3_T h^{(1)} &= \frac{+1}{2\pi}\,\frac{E_4(iT)\,E_4(iU)E_6(iU)\eta^{-24}(iU)}{j(iT)\,-\,j(iU)}\,,\\
\partial^3_U h^{(1)} &= \frac{-1}{2\pi}\,\frac{E_4(iT)E_6(iT)\eta^{-24}(iT)\,E_4(iU)}{j(iT)\,-\,j(iU)}\,.
\end{aligned} \tag{22}$$

This formula obviously determines the function $h^{(1)}(T,U)$ itself up to a polynomial Ξ that is at most quadratic in T and in U, but we are unfortunately unable to write that function in terms of familiar modular functions. However, it is easy to see that eqs. (**??**) imply

$$\partial_T\partial_U h^{(1)} \;=\; \tfrac{-1}{4\pi^2}\,\log\left(j(iT)-j(iU)\right) \;+\; \text{finite}, \tag{23}$$

The logarithmically divergent part of this one-loop gauge coupling can be also obtained by computing the one-loop gauge threshold correction for the $U(1)^2_L$ gauge group as a socalled free energy $\Delta\sim\sum\log\mathcal{M}(T,U)$ [21]. Specifically, the holomorphic masses of the momentum and winding states of the T_2 compactification are of the form

$$\mathcal{M}_{m_1,m_2,n_1,n_2}(T,U)=m_2-im_1U+in_1T-n_2UT. \tag{24}$$

Then the free energy becomes [11]

$$\Delta(T,U)=\sum_{m_1,m_2,n_1,n_2}\log(m_2-im_1U+in_1T-n_2UT)=\log(j(iT)-j(iU)), \tag{25}$$

where the sum was taken over the orbit $m_1n_1+m_2n_2=1$, and we have assumed a modular invariant regularization procedure of the infinite sum. This formula has a curious property that the coefficient of the logarithmic divergence is 1 when $T\equiv U\not\equiv 1,\rho$ but becomes 2 when $T\equiv U\equiv 1$ and 3 when $T\equiv U\equiv\rho$, in precise agreement with the number of the massive string modes that become massless in each case (respectively, 2, 4 and 6 vector multiplets). On the other hand, the correct modular transformation rules imply that $h^{(1)}_{TU}$ is not entirely modular invariant but contains a finite term, which transforms inhomogeneously under modular transformations. This fact is an indication that there does not exist an modular invariant regulator when performing the sum eq.(25), but there is a modular anomaly which spoils the modular invariance of the free energy.

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