TOPOLOGICAL REDUCTION OF 4D SYM TO 2D σ–MODELS

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ABSTRACT

By considering a (partial) topological twisting of supersymmetric Yang-Mills compactified on a 2d space with 't Hooft magnetic flux turned on we obtain a supersymmetric σ-model in 2 dimensions. For $N = 4$ SYM it maps S-duality to T-duality for σ-models on moduli space of solutions to Hitchin equations.

1. Introduction

One of the main sources of insights into the dynamics of 4 dimensional quantum field theories comes from analogies with simpler 2 dimensional quantum field theories. It is the aim of this paper to make this analogy more precise in the context of supersymmetric gauge theories in 4 dimensions and special classes of supersymmetric σ-models in 2 dimensions. In the context of $N = 4$ YM, this reduction allows us to map S-duality to T-duality of certain σ-models, thus relating electric-magnetic duality to momentum-winding duality of σ-models.

The basic idea is rather simple. We consider a Euclidean quantum field theory on a product of two Riemann surfaces $\Sigma \times C$ in the limit where the size of one of them, say C shrinks to zero. This gives rise to a quantum field theory on Σ. The reduction of 4d Yang-Mill theory to 2d is in general very complicated due to the fact that different regimes of field configurations of the 4d theory result in different 2d effective theories which are related to each other in a complicated way *. For four dimensional gauge theories a single regime of field configuration can be singled out by restricting attention to the sectors of path integral with non-trivial 't Hooft magnetic flux on C (which thus avoid having reducible gauge connections).

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* A similar problem was considered by the Verlinde brothers.[1]

Starting from supersymmetric quantum field theories in 4d, we expect to get a supersymmetric theory in 2d. This is only the case when C is a torus with periodic boundary conditions. In the case C is a torus, turning on the flux unfortunately leads to a trivial quantum field theory on Σ with all the degrees of freedom frozen out. However one can consider a topologically twisted version of the 4d theory, which does give rise to a non-trivial supersymmetric 2d theory for any choice of C with genus greater than 1. In fact we can consider a fully twisted topological theory in 4 dimensions giving rise to a topological σ-model in 2 dimensions or we can consider partial twisting of the 4 dimensional theory only along the C directions and obtain an untwisted supersymmetric σ-model on Σ. Each twisting has its virtue: The fully twisted version is useful in that the topological amplitudes in 4d, being independent of the size of C, are directly related to topological amplitudes of the σ-model in 2d. The partially twisted theory, on the other hand, even though it depends on the size of C, carries more information about non-topological aspects of the 4d theory **. We will consider both twistings in this paper.

Let us first consider $N=1$ supersymmetric theory. The manifold M^4 has a product structure and therefore the holonomy group is reduced to $U(1)_\Sigma \times U(1)_C$, where each $U(1)$ is the holonomy of the corresponding surface. The $U(1)$ charges of the supersymmetry generator is given by $(\pm\frac{1}{2}, \pm\frac{1}{2})$. In addition the supercharge carries an R charge ± 1 (which with an appropriate choice of $N=1$ theories with matter is anomaly free) which is correlated with the chirality of the spinor (even or odd number of minus signs in their $U(1)_\Sigma \times U(1)_C$ charge). If we twist the $U(1)_C$ by adding $-R/2$ to it, we find that there are two components of the supersymmetry which become spin 0 in the C direction and are both of the form $(+\frac{1}{2}, 0)$. We thus end up with a $(2,0)$ supersymmetric theory on Σ for arbitrary choice of C with genus greater than 1. If we had in addition twisted the $U(1)_\Sigma$ by adding $-R/2$ we would have obtained a topologically twisted $(2,0)$ theory in 2d. With standard twistings, in the case of $N=2$ theories the same construction leads to a $(2,2)$, and for $N=4$ it leads to a $(4,4)$ supersymmetric theory on Σ. In this paper we will mostly concentrate on the case of pure $N=4$ YM theory. Extension of these to $N=1$ and to $N=2$ theories with matter are presently under consideration.

** A possible relation between the dynamics of the 4d supersymmetric theories and those of corresponding σ-models has been conjectured in Johansen.[2]

2. Reduction

We now consider this reduction in more detail. Let us first concentrate on pure YM theory on a four-dimensional manifold M^4, which has a product structure $M^4 = \Sigma \times C$. Let us choose the metric on this manifold to be block diagonal $g = g_\Sigma \oplus g_C$, where g_Σ (g_C) is the metric on Σ (C). The YM connection can be decomposed into two pieces $A = A_\Sigma + A_C$, where A_Σ (A_C) is the component of A along Σ (C). To discuss the reduction to 2d let us rescale the metric $g_C \to \epsilon g_C$ on C. Under this transformation different terms in the action scale differently

$$S = \frac{1}{4e^2} \int_{M^4} \mathrm{Tr}\Big[\frac{1}{\epsilon} F_C \wedge * F_C + 2(d_C A_\Sigma - D_\Sigma A_C) \wedge *(d_C A_\Sigma - D_\Sigma A_C) + \epsilon F_\Sigma \wedge * F_\Sigma\Big] . \tag{2.1}$$

Operation $*$ is defined with respect to unrescaled metric $g_\Sigma \oplus g_C$ and $D_\Sigma = d_\Sigma - i[A_\Sigma, \cdot]$.

In the limit $\epsilon \to 0$ the first term in the action enforces the component A_C to be flat ($F_C = 0$), while the second term gives rise to the σ-model action. The last term produces the corrections of order $O(\epsilon)$ that are irrelevant in the limit $\epsilon \to 0$.

We will denote the moduli space of flat connections on C by $\mathcal{M}(C)$. In order to specify the flat connection A_C on $\Sigma \times C$ one should specify a map $X : \Sigma \to \mathcal{M}(C)$. In this notation the flat connection becomes

$$A_C(w, \bar{w}, z, \bar{z}) = A_C(w, \bar{w} | X(z, \bar{z})), \tag{2.2}$$

where $z, \bar{z}$ ($w, \bar{w}$) are complex coordinates on Σ (C).

The flatness condition $F_C = 0$ implies that operator D_C is nilpotent, $D_C^2 = 0$. The tangent space to the moduli space of flat connection $\mathcal{M}(C)$ is given by D_C cohomology $H^1(C, \mathcal{G})$. We will always choose representatives that satisfy harmonicity condition $D_C^\mu \alpha_\mu = 0$, which is just the gauge fixing condition. The variation of the flat connection δA_C can be decomposed with respect to some basis $\{\alpha^I\} \subset H^1(C, \mathcal{G})$ modulo the gauge transformation

$$\frac{\partial A_C}{\partial X^I} = \alpha_I + D_C E_I \tag{2.3}$$

where E defines the connection on the moduli space $\mathcal{M}(C)$ (similar construction appears in Harvey and Strominger.[3] The moduli space of flat connections $\mathcal{M}(C)$ is a Kähler manifold. It is convenient to use the complex coordinates X^i and $X^{\bar{k}}$ on $\mathcal{M}$.

The action (2.1) is essentially quadratic in A_Σ, ignoring the terms of order $O(\epsilon)$. Moreover the action does not depend on the derivatives of A_Σ with respect to the coordinates on Σ. Hence A_Σ plays the role of an auxiliary field. Therefore one can attempt to integrate out A_Σ. This can be done if the connection on C is irreducible, which would allow us to invert the Laplacian $D_{\overline{w}} D_w$ on C:

$$A_\Sigma = E_i \partial_\Sigma X^i + E_{\bar{k}} \partial_\Sigma X^{\bar{k}} \ . \tag{2.4}$$

If the gauge field on C is reducible the Laplacian has zero modes which would give rise to additional degrees of freedom on Σ (and in particular dropping the $O(\epsilon)$ terms in (2.1) cannot be justified in such cases). These additional degrees of freedom are described by residual gauge theory on Σ. Moreover if the dimension of the residual gauge symmetry jumps as we move on $\mathcal{M}$ the resulting 2d theory on Σ would be very complicated. This happens for example if we consider flat $SU(N)$ gauge fields on C. However if we consider $SU(N)/Z_N$ gauge theory and restrict the path-integral to the subsector where we turn on a non-trivial 't Hooft magnetic flux on C, then the connection on C is irreducible for all $\mathcal{M}$, the gauge group is completely broken and A_Σ can be integrated out. We will mainly concentrate on this case, but comment about some aspects of the more general case below.

Substituting the flat connection A_C and the expression for A_Σ (eq. (2.4)) into the action (2.1) one gets the σ-model action of the standard form

$$S = \frac{1}{2e^2} \int_\Sigma d^2 z \ G_{i\bar{k}} (\partial_z X^i \bar{\partial}_{\bar{z}} X^{\bar{k}} + \bar{\partial}_{\bar{z}} X^i \partial_z X^{\bar{k}}) \ . \tag{2.5}$$

It is also easy to see that turning on the θ angle for the YM is equivalent to turning on a $B-$field in the direction of the Kähler class. In this way we see that $\tau = i/4\pi e^2 + \theta/2\pi$ is now playing the role of the complexified Kähler modulus of this σ-model.

The moduli space of holomorphic instantons for this σ-model can be shown to coincide[4] with the the moduli space of self-dual connections of the 4d YM theory in the limit $\epsilon \to 0$. In particular one can view anti-self-dual connections as holomorphic connections (whose curvature vanish in the $(2,0)$ and $(0,2)$ directions) which satisfy $g^{i\bar{j}} F_{i\bar{j}} = 0$. The latter condition in the limit $\epsilon \to 0$ becomes $F_C = 0$, whereas the holomorphicity of the connection is equivalent to holomorphic instantons of the 2d theory.

Now consider the dimensional reduction of the topological YM theory which is the twisted version of the $N = 2$ $d = 4$ supersymmetric Yang-Mills theory.[5] In this case one ends up with the $(2,2)$ supersymmetric σ-model on $\mathcal{M}$. It is convenient to formulate this model in the complex notation. In the bosonic sector of $N=2$ SYM theory there is a scalar field ϕ in addition to Yang-Mills connection A. The fermionic fields are the following: a scalar η, a self-dual two form that can be decomposed to a scalar λ and $(2,0)$ and $(0,2)$ forms λ_{zw} and $\lambda_{\bar{z}\bar{w}}$, and a 1-form with the components χ_z, χ_w, $\chi_{\bar{z}}$, $\chi_{\bar{w}}$. Since the action is linear in fermionic fields λ, η, χ_z and $\chi_{\bar{z}}$, one can integrate them out. Such an integration gives rise to the following constraints: $D_w\chi_{\bar{w}}=0$, $D_{\bar{w}}\chi_w=0$, $D_w\lambda_{\bar{w}\bar{z}}=0$, $D_{\bar{w}}\lambda_{wz}=0$. These fields are cotangent to the moduli space $\mathcal{M}(C)$ of flat connections on C. Therefore in the basis $\alpha_{i\bar{w}}$, $\alpha_{\bar{k}w}$ they can be represented as linear combinations

$$\chi_{\bar{w}}=\chi^i\alpha_{\bar{w}i},\ \chi_w=\chi^{\bar{k}}\alpha_{w\bar{k}},\ \lambda_{\bar{w}\bar{z}}=\rho^i_{\bar{z}}\alpha_{\bar{w}i},\ \lambda_{wz}=\rho^{\bar{k}}_z\alpha_{w\bar{k}}, \tag{2.6}$$

where χ^i, $\chi^{\bar{k}}$, $\rho^i_{\bar{z}}$ and $\rho^{\bar{k}}_z$ are two dimensional fermionic fields on Σ. The action is also quadratic in scalar fields ϕ and $\bar{\phi}$ and does not depend on the derivatives of these fields with respect to coordinates on Σ (in the leading order $\epsilon\to 0$). Therefore one can just solve the equations of motion for ϕ and $\bar{\phi}$.

Similar to the above non-supersymmetric model we integrate over components A_Σ of the gauge connection. Integration over the field ϕ and A_Σ results in a four-fermionic interaction in the Lagrangian.

At this stage, it is already clear that we get a supersymmetric twisted σ-model on $\mathcal{M}$ (**A** model) with the standard action*

$$\begin{aligned}S=\frac{1}{e^2}\int_\Sigma \mathrm{d}^2z\ \Big[G_{i\bar{k}}(\frac{1}{2}\partial_z X^i\bar{\partial}_{\bar{z}}X^{\bar{k}}+\frac{1}{2}\bar{\partial}_{\bar{z}}X^i\partial_z X^{\bar{k}}+i\rho^{\bar{k}}_z\overline{D}_{\bar{z}}\chi^i+i\rho^i_{\bar{z}}D_z\chi^{\bar{k}})\\ -R_{i\bar{k}j\bar{l}}\rho^i_{\bar{z}}\rho^{\bar{k}}_z\chi^j\chi^{\bar{l}}\Big].\end{aligned} \tag{2.7}$$

The anomaly in the fermion number is the same for the original 4d topological theory and for the σ-model. In the case of $SU(N)$, in particular the $c_1(\mathcal{M})=Nh_2$ (where $h_2\in H^2(\mathcal{M},\mathbf{Z})$), in accord with the $U(1)$ 'ghost' number violation for the $N=2$ $SU(N)$ theory.

Consider now the dimensional reduction of the $N=4$ SYM theory. It is convenient to consider the *partially*-twisted version of this theory. In the complex notation the bosonic content of the model is the following: the gauge field A, two complex scalar bosons ϕ $(\bar{\phi})$

* The **A** twisting is inherited from four dimensions. If we consider the partial twisting of the four dimensional theory described above, we would obtain the untwisted σ-model on $\mathcal{M}$.[6]

and φ ($\bar{\varphi}$) and $(1,0)$ and $(0,1)$ forms on C denoted by ϕ_w and $\phi_{\bar{w}}$ respectively**. These non-scalar bosonic fields appear because twisting is performed with conserved current that includes a bosonic contribution. The fermionic fields are doublets with respect to the $SU(2)$ global group which is the unbroken subgroup of $SU(4)$ corresponding to $N=4$ supersymmetry (the BRST charges are doublets with respect to this global group). There are the following fermionic fields: two scalars (on C) η^a_- and λ^a_-, $(1,0)$ and $(0,1)$ forms λ^a_{w-} and $\bar{\lambda}^a_{\bar{w}-}$, two vectors represented (after contracting with metric) by χ^a_{w+}, $\bar{\chi}^a_{\bar{w}+}$, and additional scalars on C denoted by χ^a_+ and $\bar{\chi}^a_+$. Here the indices $a=1,2$ correspond to the doublet representation of the unbroken $SU(2)$ global group, the vector indices w and $\bar{w}$ correspond to the surface C, and the indices $\pm$ stand for (right-) left-handed spinors indices on Σ.

The dimensional reduction here is slightly different from that of above cases in the following respects. First, some of the bosonic fields which are scalar fields in the untwisted theory become 1-forms in the twisted model. Therefore their kinetic term is not suppressed as $\epsilon \to 0$ and may still correspond to propagating degrees of freedom in the dimensionally reduced theory. Second, there are unsuppressed terms in the Lagrangian which describe ϕ^4 interactions of the bosonic fields. In the limit $\epsilon \to 0$ the equations of motion reduce to

$$F_{w\bar{w}} = -i[\phi_{\bar{w}}, \phi_w], \quad D_w \phi_{\bar{w}} = 0, \quad D_{\bar{w}} \phi_w = 0. \tag{2.8}$$

This set of equations coincides with the Hitchin's equations for 'stable pairs'.[8] The moduli space $\mathcal{M}^H$ of solutions to Hitchin equations is the target space of supersymmetric 2d σ-model.

The totally twisted 4d Yang-Mills theory reduces to the twisted version of 2d sigma model. The twisting current has a bosonic piece and hence some of the bosonic fields become 1-forms on the world sheet Σ. In 4-dimensional theory this current generates $U(1)$ global phase rotations of ϕ and $\bar{\phi}$. In terms of the *partially* twisted theory the bosonic contributions to the current is $j_n = \mathrm{Tr}(\bar{\phi}^w\ \partial_n \phi_w - \partial_n \bar{\phi}^w\ \phi_w + ...)$, where n is a worldsheet index on Σ. Under the dimensional reduction this current becomes

$$j_n = \int_C \mathrm{Tr}(\bar{\phi}_{\bar{w}}(X) \partial_n \phi_{\bar{w}}(X) - \partial_n \bar{\phi}_{\bar{w}}(X) \phi_w(X)) + (\text{fermionic terms}), \tag{2.9}$$

where $X(z,\bar{z})$ determines the map $\Sigma \to \mathcal{M}^{\mathcal{H}}$. The fields $\bar{\phi}_{\bar{w}}$ and ϕ_w obey the Hitchin's equations and hence are functions on $\mathcal{M}^H$. In fact, in the σ-model j_n is a Noether current corresponding to the action of $U(1)$ on $\mathcal{M}^H$ given by $(A,\phi) \to (A, e^{i\theta}\phi)$. By the equations of motion of the σ-model j_n is conserved.

** The twist that we use is the partial twisting corresponding to that used in Vafa and Witten.[7]

3. N=4 Application

Now we turn to the discussion of aspects of the reduced $N=4$ YM in two dimensions. As we discussed before the two dimensional theory we have obtained is a supersymmetric σ-model on the Hitchin space $\mathcal{M}^H$ which is a hyperKähler manifold of dimension $6g-6$. Since $\mathcal{M}^H$ is a smooth hyperKähler manifold the corresponding sigma model is a superconformal theory, which is in accord with the fact that one expects the four dimensional theory to be superconformal as well.

Since the coupling constant τ of the 4d YM theory gets identified with the unique complexified Kähler class for $\mathcal{M}^H$, the Montonen-Olive conjecture[9] for the 4d N=4 YM, gets translated to the modularity properties of the topological σ-model with respect to the Kähler moduli τ. In particular for $SU(N)$, the moduli space for τ should be a fundamental domain for the subgroup $\Gamma_0(N)$ (with lower off-diagonal entry being 0 mod N) of $SL(2,\mathbf{Z})$ (note that $\Sigma \times C$ has even quadratic form on H^2). The S-duality conjecture in 4d thus gets translated to a T-duality for this 2d σ-model. However for σ-models we basically understand how T-duality may arise and thus we may be able to shed some light on the S-duality in 4d theories. We will show why the Hitchin's σ-model has T-duality. Before doing this let us see why this map of S-duality to T-duality is a resonable thing to expect.

In fact this is a natural generalization of the S-duality for the abelian $N=4$ theory: If we consider $SU(2)$ gauge group and choose the internal space $C=T^2$, with a magnetic flux turned on, the σ-model becomes trivial (i.e. the Hitchin space is just a point). However if we don't turn on the flux, as discussed before we do not get a simple reduction to a $2d$ theory as different $4d$ field configurations lead to different regimes of the reduced theory which are connected to each other in a complicated way. In one field regime which corresponds to large expectation values for ϕ, i.e. the Higgs phase, the theory reduces to a $U(1)$ gauge theory plus a σ-model on the corresponding Hitchin space which in this case is just the cotangent of the moduli of flat connection (i.e. the cotangent of the torus which characterizes the holonomy of the unbroken $U(1)$ along the T^2 modulo the Weyl action). In other words, as noted in Girardello *et al.*[10], the piece of the partition function compactified on T^2, which grows like the volume of ϕ, can be easily extracted from this complicated effective theory and is manifestly S-dual since for large ϕ the S-duality for the non-abelian theory gets mapped to S-duality for the abelian theory. In this context the field configurations which wrap around the σ-model torus get mapped to 4d field configurations where there is a magnetic flux for this unbroken $U(1)$ and the momentum modes are the dual configurations which are identified with the electrical flux of the unbroken $U(1)$. Thus the S-duality of the abelian theory gets mapped to T-duality*. Note, however, that it would be incorrect to ignore the other field configurations which make contributions to the path-integral which do not grow like the volume of ϕ. In fact it is relatively easy to see that ignoring those would lead to a Witten index for the σ-model which does not agree with that for the 4d theory (which for $SU(2)$ is 1 for the σ-model and 10 for the 4d theory[7]). Thus to make a really non-abelian test of S-duality we turn to the case where genus of C is greater than 1 and with 't Hooft magnetic flux turned on.

There is a description of $\mathcal{M}^H$ which is most suitable for us[8]: For any gauge group G, $\mathcal{M}^H$ is a fiber space over the complex space $\mathbf{C}^d$ where $d = \mathrm{dimG}(h-1)$, whose fiber is a complex torus with complex dimension d. The complex structure of the torus varies holomorphically as we move in $\mathbf{C}^d$, but the Kähler structure of the torus is fixed and can be identified with the Kähler structure of $\mathcal{M}^H$. As we move the base point we reach points where the fiber is a singular torus but the total space is still smooth. The situation is a generalization of the cosmic string solution constructed in Greene *et al.*[12], where the base was $\mathbf{C}^1$ and the fiber a complex one dimensional torus. The basic strategy there was to use adiabatic approximation, by viewing the complex moduli as massless fields in $\mathbf{C}^1$ and to construct a hyperKähler metric by adiabatically varying the complex structure but with a fixed Kähler structure of the torus. Since the Kähler moduli is fixed for each fiber, this means that the modular properties of the Kähler moduli we will obtain, as long as we can trust the adiabatic approximation, will still be the same as that for each fiber (as the massless fields corresponding to varying it are turned off). The adiabatic approximation breaks down in the regions where the fiber becomes singular–however as was the case in[12] and as is the case for Hitchin space the total space is still a smooth hyperKähler space and we thus obtain an exact $(4,4)$ superconformal theory. Even though we may not have trusted adiabatic approximation for obtaining exact solutions, we do trust it as far as symmetries are concerned. Thus the Kähler moduli τ which can be identified with that of a non-singular fiber still enjoys the same modular properties as that of each fiber. Thus to find the modular properties of the Kähler parameter τ for $\mathcal{M}^H$ we have to study the modular properties of the Kähler modulus of the fiber torus.

Let us briefly explain why $\mathcal{M}^H$ has this toroidal fiber structure. For simplicity let us concentrate on $G = SU(2)$. Let $b_{ww} = \det\phi_w = -\frac{1}{2}\mathrm{Tr}\phi_w^2$. Then by Hitchin equations

* The fact that in this context the S-duality is equivalent to the T-duality of toroidal compactification of the reduced theory has been independently noted in a recent paper.[11]

(2.8), $\overline{\partial} b_{ww} = 0$ whose solution can be identified with $\mathbf{C}^{3h-3}$, i.e. the complex $3h-3$ dimensional space. Generically a point of $\mathbf{C}^{3h-3}$ will correspond to a b_{ww} with isolated zeroes. Let us concentrate on such a solution. Away from the zeroes of b_{ww}, ϕ_w determines a $U(1)$ subspace of $SU(2)$, by the condition that $\Lambda = \phi_w/\sqrt{b}_{ww} = \pm 1$–more precisely we obtain a line bundle on the double cover $\hat{C}$ of C, which has genus $4h-3$, branched over the zeroes of b_{ww}. Away from the branch points the gauge field restricted to this $U(1)$ part is flat as follows from the fact that $\mathrm{Tr}F(\Lambda \pm 1) = 0$ because $\mathrm{Tr}F = 0$ and $\mathrm{Tr}F\phi_w = \mathrm{Tr}[\phi_w, \phi_{\overline{w}}]\phi_w = 0$. This line bundle will have delta function singularities at the branch points that can be gotten rid of by tensoring with a fixed line bundle with opposite singularity. The possible solutions to the Hitchin equation will thus give rise to flat bundles on $\hat{C}$ which are parametrized by the Jacobian of $\hat{C}$, which can be viewed as the allowed holonomies of the $U(1)$ gauge group through the cycles of $\hat{C}$. However the allowed fluxes are parametrized by the Prym subspace of the Jacobian, which is the $3h-3$ dimensional complex torus which is odd under the Z_2 involution. This is because the involution on $\hat{C}$ exchanges the line bundle with its dual. We have thus given the description of $\mathcal{M}^H$ as a toroidal fiber space over C^{3h-3}. The generalization to $SU(N)$ is straightforward, with the base space being replaced by the space of allowed holomorphic differentials $Tr\phi_w^j$, where $j = 2, ..., N$, and by the fiber being the Prym variety of an N-fold cover of C [13]. Note that the S-duality getting mapped to T-duality of this fiber torus is a very natural generalization of what appears in the abelian case discussed above. Moreover it suggests an approach to showing S-duality for the non-abelian four dimensional theory by slicing the 4d path-integral in such a way that it becomes equivalent to a family of abelian S-dualities glued together in a nice way.

To get the precise form of the duality we thus have to study the moduli space of a complex d dimensional torus. The moduli space of a $2d$ real dimensional torus is known[14] to be

$$\frac{SO(2d,2d)}{SO(2d) \times SO(2d) \times SO(2d,2d;\mathbf{Z})}$$

If we fix an integral Kähler form $k \in H^2(T^{2d};\mathbf{Z})$ on the torus and ask about the moduli of complex structures on the torus with that fixed Kähler class the answer is described as follows[15] : Let $x^i, y^i, i = 1, ..., d$ denote the coordinates of torus with periodicity 1 in each direction which are chosen so that the Kähler form can be written as

$$k = \sum_{i=1}^{d} n_i dx^i \wedge dy^i \tag{3.1}$$

where n_i are positive integers. Let D denote the $d \times d$ diagonal matrix $D = (n_1, ..., n_d)$. Let z_i be the complex coordinates of the torus. Then we can choose them so that

$$dz^i = n_i dx^i + \sum_j \Omega_{ij} dy^j \tag{3.2}$$

where Ω is a complex, symmetric $d \times d$ matrix with a positive definite imaginary part (all follow from the fact that k defined above be a positive $(1,1)$ form) . We have $k = dz^i (\frac{1}{-2iIm\Omega})_{i\bar{j}} dz^{\bar{j}}$. We are interested in how the moduli space of complex structure and the particular complexified Kähler structure (rescaling the fixed Kähler class by t plus turning on an anti-symmetric b field in the direction of the fixed Kähler form) imbed in the Narain moduli space. There is an action of symplectic group $Sp_J(2d)$ preserving the symplectic form

$$J = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

on the moduli of complex structure, and the full moduli space of complex structures is given by the quotient $Sp_J(2d)/U(d) \times Sp_J(2d; \mathbf{Z})$, where $U(d)$ rotates the z_i among themselves. Note that $Sp_J(2d)$ is equivalent (and conjugate) to the canonical group as far as they are defined over the reals, but the group $Sp_J(2d; \mathbf{Z})$ very much depends on J (for example it would have been trivial if n_i were generic real numbers).

We will now show that the relevant moduli space for our problem is split to the complex and Kähler directions, where we just described the complex part. Since the Narain moduli space is described as a group quotient, it suffices to show that the generators of the complex deformations and the particular Kähler deformation commute. Let us first work over the real numbers, in which case we can rescale coordinates so that D is replaced by the identity matrix and J has the canonical form. It is not difficult to see that the generators of the deformations are then given by

$$\text{Complex}: \qquad (\sigma_x \otimes S; 1 \otimes A)/1 \otimes A$$

$$\text{Kahler}: \qquad (t = \sigma_x \otimes 1;\ b = i\sigma_y \otimes J;\ \sigma_z \otimes J)/\sigma_z \otimes J$$

where S and A denote symmetric and anti-symmetric generators of $Sp(2d)$, and the Pauli matrices act on the (L, R) decomposition of the Narain momenta. Note that the generators of Kähler deformations commute with those of complex deformations and form the $Sp(2)$ (or $SL(2)$) group. In fact this is the maximal subgroups of $SO(2d, 2d)$ which commutes with $Sp(2d) \subset SO(2d, 2d)$. In order to find how the modular group acts on the $Sp(2)$, given its imbedding in the Narain moduli, all we have to do is to find integral points of the group generated by $\sigma_x \otimes 1, i\sigma_y \otimes J, \sigma_z \otimes J$; We also have to recall that we have rescaled coordinates so that J is in the canonical form. If we undo this rescaling and we decompose $J = \oplus J_i$ where J_i corresponds to i-th direction corresponding to n_i, we can view our $Sp(2)$ as sitting diagonally in $\otimes Sp_i(2)$ where the common moduli τ is identified as $n_i \tau_i$ in each subfactor. With no loss of generality let us assume n_i's have no common divisor (otherwise rescale the Kähler form so this is true). Let n denote the least common multiple of n_i. Then it is clear that the common intersection of all the $SL_i(2, Z)$ is generated by T and $ST^n S$ where $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$. This generates the group $\Gamma_0(n)$. We thus have the moduli space

$$\frac{Sp(2d)}{U(d) \times Sp_J(2d, \mathbf{Z})} \times \frac{Sp(2)}{U(1) \times \Gamma_0(n)}$$

Thus the Kähler moduli of the Hitchin space has $\Gamma_0(n)$ as a modular group. For $SU(N)$, all the n_i are either N or 1, corresponding to whether they are related to combinations of cycles of $\hat{C}$ which are projected to trivial or non-trivial cycles of C. So in this case $n = N$ and we recover the prediction of the S-duality that τ should belong to the fundamental domain of $\Gamma_0(N)$. In fact there is more information in the modular transformation. In particular prediction of S-duality for $\tau \to -1/\tau$ is in accord with the relation between the Hitchin spaces for $SU(N)$ vs. $SU(N)/Z_N$.

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5. References

1. H. Verlinde and E. Verlinde, "QCD at high energies and two-dimensional field theory", Preprint PUPT-1319, hep-th/9302104.
2. A. Johansen, "Infinite conformal algebras in supersymmetric theories on four manifolds", Preprint NBI-HE-94-34.
3. J. Harvey and A. Strominger, *Comm. Math. Phys.* **151** (1993) 221.

4. S. Dostoglou, D. Salomon, "Instanton homology and symplectic fixed points", preprint 1992.
5. E. Witten, *Comm. Math.Phys.* **117**, (1988) 353.
6 E .Witten, in *Essays on Mirror Manifolds*, ed. by S. T. Yau, International Press, 1992.
7. C. Vafa and E. Witten, *Nucl. Phys.* **B431** (1994) 3.
8. N. J. Hitchin, *Proc. Lond. Math. Soc.* **55** (1987) 59.
9. C. Montonen and D. Olive, *Phys. Lett.* **B72** (1977) 117.
10. L. Girardello, A. Giveon, M. Porrati, A. Zaffaroni, *Phys. Lett.* **334B** (1994) 331.
11. J. Harvey, G. Moore and A. Strominger, "Reducing S-Duality to T-Duality", Preprint hep-th/9501022.
12. B. R. Greene, A. Shapere, C. Vafa and S.-T. Yau, *Nucl. Phys.* **B337** (1990) 1.
13. R. Donagi, "Spectral covers", Preprint 1994; N. J. Hitchin, *Duke Math. J.* **54** (1987) 91-114.
14. K. S. Narain, *Phys. Lett.* **B169** (1986) 41.
15. P. Griffiths and J. Harris *Principles of Algebraic Geometry* (New York, Wiley, 1978).
16. M. Bershadky, A. Johansen, V. Sadov and C. Vafa, "Topological Reduction of 4D SYM to 2D σ–Models", Preprint HUTP-95/A004.