OPERATOR ALGEBRA OF THE $4D$ $\mathcal{W}_3$ STRING

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ABSTRACT

The noncritical $4D$ $\mathcal{W}_3$ string is a model of $\mathcal{W}_3$ gravity coupled to two free scalar fields. In this paper we discuss its BRST quantization in direct analogy with that of the $2D$ (Virasoro) string. The physical operators form a BV-algebra. We model this BV-algebra on that of the polyderivations of a commutative ring on six variables with a quadratic constraint, or, equivalently, on the BV-algebra of (polynomial) polyvector fields on the base affine space of $SL(3,\mathbb{C})$.

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The noncritical $4D$ $\mathcal{W}_3$ string is a model of $\mathcal{W}_3$ gravity coupled to two free scalar fields. In this paper we discuss its BRST quantization in direct analogy with that of the $2D$ (Virasoro) string. The physical operators form a BV-algebra. We model this BV-algebra on that of the polyderivations of a commutative ring on six variables with a quadratic constraint, or, equivalently, on the BV-algebra of (polynomial) polyvector fields on the base affine space of $SL(3,\mathbb{C})$.

1. Introduction

This paper is a brief outline of our work, over the past few years, on the $\mathcal{W}_3$ algebra: its representation theory; the corresponding semi-infinite cohomology for special modules; the operator algebra of related $\mathcal{W}_3$ string models and its BV-algebra interpretation. A more complete exposition may be found in [1].

$\mathcal{W}_3$ gravities provide interesting examples of gauge theories based on a nonlinear algebra of constraints; namely, the $\mathcal{W}_3$ algebra extension of the Virasoro algebra (see [2,3] and references therein). The nonlinearity immediately leads to a couple of obvious complications: the adjoint action of the Cartan subalgebra on $\mathcal{W}_3$ is not diagonalizable, and similarly its action on most interesting $\mathcal{W}_3$ modules will not be diagonalizable; moreover, the tensor product of two $\mathcal{W}_3$ modules does not, in general, carry the structure of a $\mathcal{W}_3$ module. But still, the algebraic structure allows a definition of certain $\mathcal{W}_3$ gravity models through BRST quantization even though the associated $\mathcal{W}$-geometry is not yet well understood in general. In fact, working by analogy with ordinary two-dimensional gravity, there exists a well-motivated BRST quantization for $\mathcal{W}_3$ gravity coupled to conformal matter with a restricted range for the central charge of the matter CFT. The corresponding BRST charge was introduced in [4].

Here we will review the special case of $c^M = 2$. In the corresponding string interpretation the matter scalar fields would embed the world sheet of the string into a two-dimensional space-time. But, moreover, since this is a non-critical theory there are dynamical gravitational degrees of freedom – under the DDK-type ansatz these are described by a pair of scalar fields of "wrong sign" with a background charge, the so-called Liouville sector. Thus, in this string language, the model describes a $(2+2)$-dimensional string in non-trivial background fields. We call it the $4D$ $\mathcal{W}_3$ string.

More specifically, in this paper we present a geometric model for the operator algebra of the $4D$ $\mathcal{W}_3$ string, following the corresponding analysis of the $2D$ (Virasoro) string [5–7]. We begin with a summary of the main features of the $2D$ string, in language which easily generalizes to the $\mathcal{W}_3$ case. We then summarize that generalization. An appendix reviews the notions of BV-algebras and their modules which are required in this study.

1.1. Notation

In the following, $P(\mathfrak{sl}_N)$ and $Q(\mathfrak{sl}_N)$ denote the $\mathfrak{sl}_N$ weight lattice and root lattice, respectively, while $P_+(\mathfrak{sl}_N)$ denotes the dominant integral weights. The matter and Liouville sectors will be distinguished by a superscript M or L, respectively. $L(\mathfrak{sl}_N) \subset (P(\mathfrak{sl}_N) \times P(\mathfrak{sl}_N))$ is the sublattice defined by

$$L(\mathfrak{sl}_N) = \{(\Lambda^M, -i\Lambda^L) \in P(\mathfrak{sl}_N) \times P(\mathfrak{sl}_N) \mid \Lambda^M + i\Lambda^L \in Q(\mathfrak{sl}_N)\} . \tag{1.1}$$

$W(\mathfrak{sl}_N)$ is the Weyl group, ρ always denotes the corresponding principal Weyl vector, $\ell(w)$ is the length of $w \in W(\mathfrak{sl}_N)$, and w_0 is the reflection in the highest root. Further, F^{gh} is the Fock space of the ghost system required for the BRST complex of $\mathcal{W}_N$-gravity; *i.e.*, the $j = 2, \ldots, N$ (bc)-systems, $(b^{[j]}, c^{[j]})$. Throughout this paper, we will denote the zero modes of the spin-2 (Virasoro) ghosts by b_0, c_0.

2. The BV-algebra of the $2D$ $\mathcal{W}_2$ string

We first consider the BV-algebra of two-dimensional gravity coupled to $c^M = 1$ matter. The reader should consult [8,9,5,6,10], and especially [7] and the talk by Zuckerman in these proceedings, for additional discussion and detailed proofs.

2.1. The cohomology problem

A Fock space representation of the Virasoro algebra, $F(\Lambda, \alpha_0)$, is parametrized by the momentum, Λ, of the underlying Fock space of a single scalar field with background charge α_0. If we interpret Λ as an $\mathfrak{sl}_2$ weight then this representation has highest (Virasoro) weight $h = \frac{1}{2}(\Lambda, \Lambda + 2\alpha_0\rho)$, and the central charge is given by $c = 1 - 6\alpha_0^2$.

An important problem in the study of $2D$ Virasoro string is to compute the BRST cohomology of the tensor product $F(\Lambda^M,0)\otimes F(\Lambda^L,2i)$ of Fock spaces with $c=1$ and $c=25$, respectively. More precisely, we will here introduce the VOA, $\mathfrak{C}$, corresponding to $\bigoplus_{(\Lambda^M,-i\Lambda^L)\in L(\mathfrak{sl}_2)} F(\Lambda^M,0)\otimes F(\Lambda^L,2i)\otimes F^{gh}$. The BRST complex lifts to $\mathfrak{C}$, on which the differential d acts as the charge of a spin-1 current. We will denote by $\mathfrak{H}_{\mathcal{W}_2}\equiv H(\mathcal{W}_2,\mathfrak{C})$ the cohomology of the complex $(\mathfrak{C},d)$. For the Virasoro case it is useful to define a relative cohomology, $H_{rel}(\mathcal{W}_2,\mathfrak{C})$, for the subcomplex $(\mathfrak{C},d)_{rel}$ of operators annihilated by b_0 and $L_0^{tot}\equiv\{b_0,d\}$.

The vertex operator realization of $\widehat{\mathfrak{sl}_2}$, together with the Liouville momentum operator, $-ip^L$, give rise to an $\mathfrak{sl}_2\oplus\mathfrak{u}_1$ symmetry on $\mathfrak{C}$ that commutes with the BRST operator and yields a decomposition of the cohomology, $H(\mathcal{W}_2,\mathfrak{C})$, into a direct sum of finite dimensional irreducible modules. The complete description of the cohomology space is given by the following theorem due to [8,9,5].

Theorem 2.1. *The cohomology $H(\mathcal{W}_2,\mathfrak{C})$ is isomorphic, as an $\mathfrak{sl}_2\oplus\mathfrak{u}_1$ module, to the direct sum of doublets of irreducible $\mathfrak{sl}_2\oplus\mathfrak{u}_1$ modules with highest weights in a set of disjoint lines $\{(\Lambda,\Lambda')+(\lambda,w^{-1}\lambda)\,|\,\lambda\in P_+(\mathfrak{sl}_2)\}$ labeled by $w\in W(\mathfrak{sl}_2)=\{1,r\}$; i.e.,*

$$H^n_{rel}(\mathcal{W}_2,\mathfrak{C})\;\cong\;\bigoplus_{w\in W(\mathfrak{sl}_2)}\;\bigoplus_{(\Lambda,\Lambda')\in S^n_w}\;\bigoplus_{\lambda\in P_+(\mathfrak{sl}_2)}\mathcal{L}(\Lambda+\lambda)\otimes\mathbb{C}_{\Lambda'+w^{-1}\lambda}\,, \tag{2.1}$$

where the nontrivial sets of cone tips at $gh=n$, S^n_w, are:

$$\begin{aligned}
n=0:&\qquad S^0_1=\{(0,0)\}\\
n=1:&\qquad S^1_1=\{(\rho,-\rho)\}\,,\quad S^1_r=\{(0,-2\rho)\}\\
n=2:&\qquad S^2_r=\{(0,-4\rho)\}\,.
\end{aligned}$$

Remarks:

i. The doublet structure (note "doublet" does *not* refer to an $\mathfrak{sl}_2$ representation!) follows from $H^\bullet(\mathcal{W}_2,\mathfrak{C})\cong H^\bullet_{rel}(\mathcal{W}_2,\mathfrak{C})\oplus H^{\bullet-1}_{rel}(\mathcal{W}_2,\mathfrak{C})$

ii. The Fock space cohomology, $H(\mathcal{W}_2,F(\Lambda^M,0)\otimes F(\Lambda^L,2i))$, may be determined either directly (see, *e.g.*, [9]), or by decomposing $F(\Lambda^M,0)$ into irreducible modules and then computing $H_{rel}(\mathcal{W}_2,L(\Lambda,0)\otimes F(\Lambda^L,2i))$ (see [8]).

2.2. The BV-algebra and structure theorems

The operator cohomology $(\mathfrak{H}_{\mathcal{W}_2},\,\cdot\,,b_0)$ forms a BV-algebra [7], where the dot product is just the normal-ordered product of operators in cohomology. The ground ring $\mathfrak{H}^0_{\mathcal{W}_2}$ is isomorphic to the algebra of polynomial functions on the complex plane; *i.e.*, $\mathfrak{H}^0_{\mathcal{W}_2}\cong\mathcal{C}_2$, where $\mathcal{C}_2=\mathbb{C}[x_1,x_2]$ is the free Abelian algebra on two generators.

We denote the corresponding generators of the ground ring by $\widehat{x}_i$, $i = 1, 2$. It is shown in the appendix that a natural BV-algebra associated with the ring $\mathcal{C}_2$ is $(\mathcal{P}(\mathcal{C}_2), \cdot, \Delta_S)$, where $\mathcal{P}(\mathcal{C}_2)$ is the space of polyderivations of $\mathcal{C}_2$. One is led to ask precisely how these two BV-algebras are related, and this is answered by the following structure theorem.

Theorem 2.2 [7].

i. *There is a natural map* $\pi : \mathfrak{H}^\bullet_{\mathcal{W}_2} \longrightarrow \mathcal{P}^\bullet(\mathcal{C}_2)$ *which is a BV-algebra homomorphism onto the BV-algebra of polyderivations* $(\mathcal{P}(\mathcal{C}_2), \cdot, \Delta_S)$.
ii. *There exists an embedding* $\imath : \mathcal{P}(\mathcal{C}_2) \longrightarrow \mathfrak{H}_{\mathcal{W}_2}$ *that preserves the dot product and satisfies* $\pi \circ \imath = \mathrm{id}$.

Remark: The projection, π, is defined by induction on the degree, n. For $n = 0$, π is the isomorphism discussed above, *i.e.*, $\pi(\widehat{x}_i) = x_i$, $i = 1, 2$. It is then extended to $n > 0$ using the condition (for any $a \in \mathfrak{H}^n_{\mathcal{W}_2}$ and $x \in \mathcal{C}_2$)

$$\pi(a)(x) \;=\; \pi([a, \widehat{x}])\,. \tag{2.2}$$

This theorem is a neat summary of the discussions of [5,6] on the geometric interpretation of a part of the cohomology. One may proceed to a detailed understanding of the full cohomology. Let us denote $(\mathfrak{H}_{\mathcal{W}_2})_+ \equiv \imath(\mathcal{P}(\mathcal{C}_2))$ and $(\mathfrak{H}_{\mathcal{W}_2})_- \equiv \operatorname{Ker} \pi$, then $\mathfrak{H}_{\mathcal{W}_2} \cong (\mathfrak{H}_{\mathcal{W}_2})_+ \oplus (\mathfrak{H}_{\mathcal{W}_2})_-$. There is a unique element in $(\mathfrak{H}_{\mathcal{W}_2})_+$, $\widehat{\Omega}$ of ghost number two at the weight $(0, -2\rho)$, for which $b_0 \widehat{\Omega} \notin \imath(\mathcal{P}(\mathcal{C}_2))$. By studying the action of $\mathfrak{H}_{\mathcal{W}_2}$ on its BV-ideal $(\mathfrak{H}_{\mathcal{W}_2})_-$, Lian and Zuckerman [7] conclude that the BV-algebra $(\mathfrak{H}_{\mathcal{W}_2}, \,, \cdot, b_0)$ is generated by 1, the ground ring generators $\widehat{x}_i$ and $\widehat{\Omega}$. The polyderivation $\Omega = \pi(\widehat{\Omega})$ is the unique nontrivial homology class of the BV-operator Δ_S. Thus, the "gluing" of $(\mathfrak{H}_{\mathcal{W}_2})_+$ and $(\mathfrak{H}_{\mathcal{W}_2})_-$, accomplished by the BV-operator b_0, is underlined by a simple algebraic principle:

The BV-algebra $(\mathfrak{H}_{\mathcal{W}_2}, \cdot, b_0)$ is the minimal BV-algebra extension of $(\mathcal{P}(\mathcal{C}_2), \cdot, \Delta_S)$ in which the cohomology of the BV-operator b_0 is trivial.

There is also a nice geometric understanding for the BV-module $(\mathfrak{H}_{\mathcal{W}_2})_-$. Introduce the "dual" ring module M_r as in the appendix. It is shown in [7] that $(\mathfrak{H}_{\mathcal{W}_2})^1_-$ is isomorphic as a ground ring module to M_r. In the appendix we see that there is a natural BV-module of $\mathcal{P}(\mathcal{C}_2)$ whose lowest grade space is M_r: namely, the twisted polyderivations $\mathcal{P}(\mathcal{C}_2, M_r)$.

Theorem 2.3. *There is a natural map* $\pi' : (\mathfrak{H}_{\mathcal{W}_2})^\bullet_- \to \mathcal{P}^{\bullet-1}(\mathcal{C}_2, M_r)$ *which is a BV-morphism of BV-modules.*

Remark: The projection, π', is again defined by induction on the degree, using the isomorphism discussed above for $n = 1$ and extending to $n > 1$ using (2.2).

3. The BV-algebra of the $4D$ $\mathcal{W}_3$ string

In this section we will simply list the appropriate changes which distinguish the corresponding results for the $\mathcal{W}_3$ string, for which the underlying problem is the computation of the BRST cohomology of the tensor product $F(\Lambda^M,0)\otimes F(\Lambda^L,2i)$ of Fock spaces with $c^M=2$ and $c^L=98$. Many of these differences can be traced to the nature of the $\mathcal{W}_3$ algebra and its representation theory which was briefly touched on in the introduction. For a complete discussion, the reader should consult [1].

The immediate difference is that the $\mathfrak{sl}_2$ structure apparent in the Virasoro case is replaced by $\mathfrak{sl}_3$. In particular, a Fock space representation of the $\mathcal{W}_3$ algebra is now labeled by two components of momentum, which can be identified with an $\mathfrak{sl}_3$ weight. We again lift the cohomology problem to the VOA, $\mathfrak{C}$, corresponding to $\bigoplus_{(\Lambda^M,-i\Lambda^L)\in L(\mathfrak{sl}_3)} F(\Lambda^M,0)\otimes F(\Lambda^L,2i)\otimes F^{gh}$. The cohomology of the complex $(\mathfrak{C},d)$ is now denoted by $\mathfrak{H}_{\mathcal{W}_3}\equiv H(\mathcal{W}_3,\mathfrak{C})$. The vertex operator realization of $\widehat{\mathfrak{sl}_3}$ on the $c=2$ Fock spaces, together with the two Liouville momenta, provide a realization of $\mathfrak{sl}_3\oplus(\mathfrak{u}_1)^2$ on $\mathfrak{C}$.

Given that there are now two antighost zero modes, $b_0=b_0^{[2]}$ and $b_0^{[3]}$, it might seem natural that the cohomology should display a quartet rather than doublet structure. This expectation is in fact realized. We call the lowest ghost number state in a given quartet the "prime" state, and denote by $H_{\rm pr}(\mathcal{W}_3,\mathfrak{C})$ the space of prime states. However, the situation is actually rather subtle – in particular, it turns out that $b_0^{[3]}$ is not well-defined on the cohomology. Thus, although relative cohomology can be defined it is not obviously useful – the nondiagonalizability of the operator $W_0^{tot}\equiv\{b_0^{[3]},d\}$ makes it extremely difficult to analyze. As a result, unlike the doublets of the Virasoro case, the quartet decomposition is at the level of vector spaces only, and there is no obvious intrinsic characterization of prime states as specific cohomology classes.

Finally, the cohomology will lie in cones, rather than lines, now labeled by the Weyl group $W(\mathfrak{sl}_3)$. In place of Theorem 2.1 we therefore have

Theorem 3.1. *The cohomology $H(\mathcal{W}_3,\mathfrak{C})$ is isomorphic, as an $\mathfrak{sl}_3\oplus(\mathfrak{u}_1)^2$ module, to the direct sum of quartets of irreducible $\mathfrak{sl}_3\oplus(\mathfrak{u}_1)^2$ modules with the highest weights in a set of disjoint cones $\{(\Lambda,\Lambda')+(\lambda,w^{-1}\lambda)\,|\,\lambda\in P_+(\mathfrak{sl}_3),\,(\Lambda,\Lambda')\in\mathcal{S}_w\}$ labeled by $w\in W(\mathfrak{sl}_3)$; i.e.,*

$$H^n_{\rm pr}(\mathcal{W}_3,\mathfrak{C})\;\cong\;\bigoplus_{w\in W(\mathfrak{sl}_3)}\;\bigoplus_{(\Lambda,\Lambda')\in\mathcal{S}^n_w}\;\bigoplus_{\lambda\in P_+}\left(\mathcal{L}(\Lambda+\lambda)\otimes\mathbb{C}_{\Lambda'+w^{-1}\lambda}\right)\,. \tag{3.1}$$

where the sets $\mathcal{S}^n_w$ (tips of the cones) are listed in Table 3.1, and

$$H^n\;\cong\;H^n_{pr}\oplus H^{n-1}_{pr}\oplus H^{n-1}_{pr}\oplus H^{n-2}_{pr}\,. \tag{3.2}$$

n	w	(Λ, Λ')
0	1	$(0, 0)$
1	1	$(\Lambda_2, \Lambda_1 - \Lambda_2)$, $(\Lambda_1 + \Lambda_2, 0)$, $(\Lambda_1, -\Lambda_1 + \Lambda_2)$
	r_1	$(0, -2\Lambda_1 + \Lambda_2)$
	r_2	$(0, \Lambda_1 - 2\Lambda_2)$
2	1	$(2\Lambda_2, -\Lambda_2)$, $(0, -\Lambda_1 - \Lambda_2)$, $(2\Lambda_1, -\Lambda_1)$
	r_1	$(\Lambda_1, -2\Lambda_1)$, $(\Lambda_2, -3\Lambda_1 + \Lambda_2)$, $(0, -4\Lambda_1 + 2\Lambda_2)$
	r_2	$(\Lambda_2, -2\Lambda_2)$, $(\Lambda_1, \Lambda_1 - 3\Lambda_2)$, $(0, 2\Lambda_1 - 4\Lambda_2)$
	r_{12}	$(0, -3\Lambda_2)$
	r_{21}	$(0, -3\Lambda_1)$
3	1	$(\Lambda_1 + \Lambda_2, -\Lambda_1 - \Lambda_2)$
	r_1	$(\Lambda_2, -2\Lambda_1 - \Lambda_2)$, $(\Lambda_1, -4\Lambda_1 + \Lambda_2)$, $(\Lambda_2, -5\Lambda_1 + 2\Lambda_2)$
	r_2	$(\Lambda_1, -\Lambda_1 - 2\Lambda_2)$, $(\Lambda_2, \Lambda_1 - 4\Lambda_2)$, $(\Lambda_1, 2\Lambda_1 - 5\Lambda_2)$
	r_{12}	$(\Lambda_2, -\Lambda_1 - 3\Lambda_2)$, $(0, \Lambda_1 - 5\Lambda_2)$, $(\Lambda_2, -5\Lambda_2)$
	r_{21}	$(\Lambda_1, -3\Lambda_1 - \Lambda_2)$, $(0, -5\Lambda_1 + \Lambda_2)$, $(\Lambda_1, -5\Lambda_1)$
	r_3	$(0, -2\Lambda_1 - 2\Lambda_2)$
4	r_1	$(0, -4\Lambda_1 - \Lambda_2)$
	r_2	$(0, -\Lambda_1 - 4\Lambda_2)$
	r_{12}	$(\Lambda_2, -2\Lambda_1 - 4\Lambda_2)$, $(\Lambda_1, -\Lambda_1 - 5\Lambda_2)$, $(0, -6\Lambda_2)$
	r_{21}	$(\Lambda_1, -4\Lambda_1 - 2\Lambda_2)$, $(\Lambda_2, -5\Lambda_1 - \Lambda_2)$, $(0, -6\Lambda_1)$
	r_3	$(0, -3\Lambda_1 - 3\Lambda_2)$, $(2\Lambda_1, -4\Lambda_1 - 3\Lambda_2)$, $(2\Lambda_2, -3\Lambda_1 - 4\Lambda_2)$
5	r_{12}	$(0, -2\Lambda_1 - 5\Lambda_2)$
	r_{21}	$(0, -5\Lambda_1 - 2\Lambda_2)$
	r_3	$(\Lambda_1, -5\Lambda_1 - 3\Lambda_2)$, $(\Lambda_1 + \Lambda_2, -4\Lambda_1 - 4\Lambda_2)$, $(\Lambda_2, -3\Lambda_1 - 5\Lambda_2)$
6	r_3	$(0, -4\Lambda_1 - 4\Lambda_2)$

Table 3.1. The sets $\mathcal{S}_w^n$

Remarks:

i. The Fock space cohomology, $H(\mathcal{W}_3, F(\Lambda^M, 0) \otimes F(\Lambda^L, 2i))$, has not been determined directly. We have computed it [11–13,1] for $-i\Lambda^L + 2\rho \in P_+(\mathfrak{sl}_3)$ by decomposing the Fock space $F(\Lambda^M, 0)$ into irreducible modules, and then computing $H(\mathcal{W}_3, L(\Lambda, 0) \otimes F(\Lambda^L, 2i))$ using (generalized) Verma module resolutions. The latter cohomology displays qualitatively new features, which, in this particular case, can be explained by the more complicated submodule structure of Verma modules of a higher rank $\mathcal{W}$-algebra.

ii. Let $\mathfrak{C}_w \subset \mathfrak{C}$ denote the subspace with shifted Liouville momentum $-i\Lambda^L + 2\rho \in w^{-1}P_+(\mathfrak{sl}_3)$. In the Virasoro case it is enough to know the cohomology for $\mathfrak{C}_1$, the fundamental Weyl chamber, since the remainder is related by a nondegenerate pairing (duality). For the $\mathcal{W}_3$ case there is again such a pairing,

$$H^\bullet(\mathcal{W}_3, F(\Lambda^M, 0) \otimes F(\Lambda^L, 2i)) \;\cong\; H^{8-\bullet}(\mathcal{W}_3, F(\Lambda^M, 0) \otimes F(w_0 \cdot \Lambda^L, 2i))\,, \quad (3.3)$$

for all $(\Lambda^M, -i\Lambda^L) \in L$, where $w_0 \cdot \Lambda^L = w_0(\Lambda^L + 2i\rho) - 2i\rho$, but it only relates the cohomology in $\mathfrak{C}_w$ and $\mathfrak{C}_{w_0 w}$ and thus is not sufficient to deduce the whole cohomology from $H(\mathcal{W}_3, \mathfrak{C}_1)$. The Fock space cohomology for $-i\Lambda^L + 2\rho$ in the other Weyl chambers has been derived from the assumption of a kind of Weyl group symmetry [1].

We may now study $(\mathfrak{H}_{\mathcal{W}_3}, \cdot, b_0)$ as a BV-algebra. The ground ring $\mathfrak{H}^0_{\mathcal{W}_3}$ in this case is the associative Abelian algebra generated by the identity operator 1, together with the $\mathfrak{sl}_3$ triplet and anti-triplet operators $\widehat{x}_\sigma$ and $\widehat{x}^\sigma$, $\sigma = 1, 2, 3$, subject to the constraint

$$\widehat{x}_\sigma \cdot \widehat{x}^\sigma \;=\; 0\,. \quad (3.4)$$

Thus we see that $\mathfrak{H}^0_{\mathcal{W}_3}$ is isomorphic to the ring $\mathcal{R}_3$ discussed in the appendix, and this allows us to present a geometric model for $\mathfrak{H}_{\mathcal{W}_3}$. Define the natural map π between $(\mathfrak{H}_{\mathcal{W}_3}, \cdot, b_0)$ and $(\mathcal{P}(\mathcal{R}_3), \cdot, \Delta_S)$ using (2.2).

Theorem 3.2 [1].

i. The map $\pi : \mathfrak{H}^\bullet_{\mathcal{W}_3} \to \mathcal{P}^\bullet(\mathcal{R}_3)$ is a BV-algebra homomorphism.

ii. $\mathfrak{I} \equiv \operatorname{Ker} \pi$ is a BV-ideal of $\mathfrak{H}_{\mathcal{W}_3}$. The exact sequence of BV-algebras

$$0 \;\longrightarrow\; \mathfrak{I} \;\longrightarrow\; \mathfrak{H}_{\mathcal{W}_3} \;\stackrel{\pi}{\longrightarrow}\; \mathcal{P}(\mathcal{R}_3) \;\longrightarrow\; 0\,. \quad (3.5)$$

splits as a sequence of $\imath(\mathcal{P}(\mathcal{R}_3))$ dot modules, where $\imath : \mathcal{P}(\mathcal{R}_3) \to \mathfrak{H}_{\mathcal{W}_3}$ is a dot algebra homomorphism such that $\pi \circ \imath = \mathrm{id}$.

iii. The cohomology of b_0 on $\mathfrak{H}_{\mathcal{W}_3}$ is trivial.

The last statement implies that the BV-algebra of cohomology is an extension of the BV-algebra of polyderivations in a way somewhat similar to the Virasoro case.

However, the kernel $\mathfrak{I}$ is now rather large, containing, in particular, most operators in $H(\mathcal{W}_3, \mathfrak{C}_w)$, $w \neq 1$. Further, unlike in the Virasoro case, the bracket and product on $\mathfrak{I}$ are nontrivial. Moreover, as seen above, the pairing (3.3) no longer determines the remaining structure. Thus it is important to obtain the analogue of Theorem 2.3. As discussed in the appendix, there are now 6 natural ring module structures M_w, labeled by the Weyl group. For a given $\mathfrak{C}_w$, $w \in W(\mathfrak{sl}_3)$, the smallest ghost number with nontrivial cohomology is $\ell(w)$ where precisely one cone appears, $\widehat{M}_w$, which is isomorphic as a ring module with M_w [1]. In fact, there is a natural map from $\mathfrak{I}$ to twisted polyderivations, which we denote by π_w,

$$\pi_w(\Phi)(x_{i_1}, \ldots, x_{i_n}) \;=\; \pi_w([\ldots [[\Phi, \widehat{x}_{i_1}], \ldots], \widehat{x}_{i_n}]) \,, \tag{3.6}$$

that identifies $\widehat{M}_w$ and M_w, and maps $\Phi \in \mathfrak{H}_{\mathcal{W}_3}^{\ell(w)+n}$, with $-i\Lambda^L + 2\rho$ sufficiently deep inside $w^{-1}P_+(\mathfrak{sl}_3)$, onto a twisted polyderivation $\pi_w(\Phi) \in \mathcal{P}^n(\mathcal{R}_3, M_w)$

Thus, sufficiently far from the overlaps of the Weyl chambers of shifted Liouville momentum, the cohomology can be identified with the twisted polyderivations of $\mathcal{R}_3$. It remains to characterize just how the different regions are patched together. Since the kernel $\mathfrak{I}$ is a BV-ideal of $\mathfrak{H}_{\mathcal{W}_3}$, it is a BV-module and therefore a G-module. Precise calculations show that $\mathfrak{I}^n = 0$ for $n < 1$, and $\mathfrak{I}^1 \cong \widehat{M}_{r_1} \oplus \widehat{M}_{r_2}$. As before, we may construct the natural map $\pi' \equiv \pi_{r_1} \oplus \pi_{r_2} : \mathfrak{I}^n \to \mathcal{P}^{n-1}(\mathcal{R}_3, M_{r_1}) \oplus \mathcal{P}^{n-1}(\mathcal{R}_3, M_{r_2})$, which is equal to the identity on $\mathfrak{I}^1$ and for $n \geq 2$ is given by the multiple brackets (3.6).

Theorem 3.3 [1]. *The map π' is a G-morphism of G-modules.*

We conjecture that this holds at the level of BV-modules. The remainder of the structure of the cohomology is now determined by duality. For details, the reader should consult [1].

The geometrical aspects of the cohomology are more manifest if we realize $\mathcal{P}(\mathcal{R}_3)$ as the space of regular polyvector fields on the base affine space of $SL(3,\mathbb{C})$.

Following [14], the base affine space of $SL(3,\mathbb{C})$ is defined as the quotient $A = N_+ \backslash SL(3,\mathbb{C})$, where N_+ is the nilpotent subroup generated by the positive root generators. The space of regular functions on A, $\mathcal{E}(A)$, consists of those functions in $\mathcal{E}(G)$ that are invariant under[1] N_+^L, and carries a representation of $(\mathfrak{sl}_3)_R \oplus (\mathfrak{u}_1^2)_L$. It is an immediate consequence of the Peter-Weyl theorem that $\mathcal{E}(A)$ is a model space for $\mathfrak{sl}_3$,

$$\mathcal{E}(A) \;\cong \bigoplus_{\Lambda \in P_+(\mathfrak{sl}_3)} (\mathcal{L}(\Lambda) \otimes \mathbb{C}_{\Lambda^*}) \,. \tag{3.7}$$

[1] We will distinguish between left and right actions by labels L and R.

The space of regular polyvector fields on the base affine space, $\mathcal{P}(A)$, is shown to be a BV-algebra – isomorphic to $\mathcal{P}(\mathcal{R}_3)$ – in [1]. It can be hoped that this reinterpretation will lead to new insights into the geometry of $\mathcal{W}$-algebras.

Appendix A. G-algebras, BV-algebras, and their modules

A.1. Preliminaries

In this appendix we will be discussing algebras $(\mathfrak{A}, \cdot)$ which are $\mathbb{Z}$-graded, supercommutative and associative. (Prefixes such as super or graded will generally be implicitly understood throughout.) Thus $\mathfrak{A} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{A}^n$, with degree $|a| = n$ for $a \in \mathfrak{A}^n$, and the product $\cdot : \mathfrak{A}^m \times \mathfrak{A}^n \to \mathfrak{A}^{m+n}$ obeys, $a \cdot b = (-1)^{|a||b|} b \cdot a$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for any homogeneous $a, b, c \in \mathfrak{A}$.

Let us call $D : \mathfrak{A}^m \to \mathfrak{A}^{m+|D|}$ a 1st-order derivation of degree $|D|$ if

$$D(a \cdot b) = D(a) \cdot b + (-1)^{|a||D|} a \cdot D(b)\,. \tag{A.1}$$

We will refer to (A.1) as the Leibniz rule, and will denote by $\mathcal{D}(\mathfrak{A}, M)$ (or simply $\mathcal{D}(\mathfrak{A})$ if $M = \mathfrak{A}$) the space of derivations of $(\mathfrak{A}, \cdot)$ with values in an $\mathfrak{A}$-module, M. There are two distinct generalizations of the notion of a derivation which will play a role in the following:

i. First let us define the operation of left multiplication by $a \in \mathfrak{A}$, $l_a : \mathfrak{A} \to \mathfrak{A}$, via $l_a(b) = a \cdot b$. Then (A.1) is clearly equivalent to the statement that for all $a \in \mathfrak{A}$,

$$D l_a - (-1)^{|a||D|} l_a D - l_{Da} = 0\,. \tag{A.2}$$

We may then generalize by induction, defining the map $D : \mathfrak{A} \to \mathfrak{A}$ to be an nth-order derivation of degree $|D|$ if $D l_a - (-1)^{|a||D|} l_a D - l_{Da}$ is an $(n-1)$th-order derivation (see, *e.g.*, [15–17]).

ii. A second generalization – which we only introduce for an $\mathcal{R}$-module M, where $(\mathcal{R}, \cdot)$ is an Abelian algebra [18] – is the space $\mathcal{P}^n(\mathcal{R}, M)$ of polyderivations of order n with coefficients in M, defined inductively by $\mathcal{P}^0(\mathcal{R}, M) \cong M$, $\mathcal{P}^1(\mathcal{R}, M) \cong \mathcal{D}(\mathcal{R}, M)$, and $\mathcal{P}^n(\mathcal{R}, M)$, $n \geq 2$, is the space of those $a \in \mathcal{D}(\mathcal{R}, \mathcal{P}^{n-1})$ that satisfy $a(x, y) = -a(y, x)$, $x, y \in \mathcal{R}$, where $a(x, y)$ denotes the element $a(x)(y) \in \mathcal{P}^{n-2}(\mathcal{R}, M)$.

A.2. Definition of G- and BV-algebras

A G-algebra [19], $(\mathfrak{A}, \cdot, [-,-])$, is defined as a dot algebra with the additional structure of a $\mathbb{Z}$-graded Lie algebra under the bracket operation, $[-,-] : \mathfrak{A}^m \times \mathfrak{A}^n \to$

$\mathfrak{A}^{m+n-1}$, where $[a,b] = -(-1)^{(|a|-1)(|b|-1)}[b,a]$, such that the bracket acts as a derivation of the algebra. It is clear from the definitions above that for any G-algebra the subspace $\mathfrak{A}^0$ is an Abelian algebra with respect to the dot product. Similarly, $\mathfrak{A}^1$ is a Lie algebra with respect to the bracket.

In contrast, a BV-algebra [20,15,21] $(\mathfrak{A}, \cdot, \Delta)$ is a dot algebra with the additional structure of a second order derivation Δ (BV-operator) of degree -1 satisfying $\Delta^2 = 0$.

There is a close relation between the two classes of algebras. Indeed, for any BV-algebra $(\mathfrak{A}, \cdot, \Delta)$, the bracket

$$[a,b] \; = \; (-1)^{|a|}\Big(\Delta(a\cdot b) - (\Delta a)\cdot b - (-1)^{|a|}a\cdot(\Delta b)\Big)\,, \qquad a,b\in\mathfrak{A}\,, \tag{A.3}$$

introduces on $\mathfrak{A}$ the structure of a G-algebra.

A.3. Definition of G- and BV-modules

We may introduce the notion of a G- or BV-module by generalizing the dot and bracket action of the G- and BV-algebras on themselves. Let $\mathfrak{M} = \bigoplus_{n\in\mathbb{Z}}\mathfrak{M}^n$ be a $\mathbb{Z}$-graded module of $(\mathfrak{A}, \cdot)$. If $\mathfrak{A}$ has the additional structure of a BV-algebra, then we define $\mathfrak{M}$ to be a BV-module if there further exists a map $\Delta_M : \mathfrak{M}^n \to \mathfrak{M}^{n-1}$, a second order derivation of the dot action of $\mathfrak{A}$ on $\mathfrak{M}$, such that $\Delta_M^2 = 0$. In that case we may define a bracket operation, $[-,-]_M : \mathfrak{A}^m \times \mathfrak{M}^n \to \mathfrak{M}^{m+n-1}$, by

$$[a,m]_M \; = \; (-1)^{|a|}\big(\Delta_M(a\cdot m) - (\Delta a)\cdot m - (-1)^{|a|}a\cdot(\Delta_M m)\big)\,, \tag{A.4}$$

for $a\in\mathfrak{A}\,,\ m\in\mathfrak{M}$. This bracket satisfies

$$[a\cdot b, m]_M \; = \; a\cdot[b,m]_M + (-1)^{|a||b|}b\cdot[a,m]_M\,. \tag{A.5}$$

Moreover, the operators $[a,-]_M$, $a\in\mathfrak{A}$, define a representation of the Lie algebra $(\mathfrak{A},[-,-])$ and act as derivations of the dot action of $\mathfrak{A}$ on $\mathfrak{M}$. Any dot module $\mathfrak{M}$ for which a bracket operation with these properties exists will be called a G-module. Clearly then, a BV-module is automatically a G-module.

A.4. General examples

A standard geometric example of a dot algebra is that of smooth polyvector fields on a given manifold, together with the wedge product. The Lie bracket on vector fields and functions may be extended by induction to all polyvectors. The resulting structure is a G-algebra. A straightforward abstraction of this example is the space $\mathcal{P}(\mathcal{R}) = \bigoplus_{n\geq 0}\mathcal{P}^n(\mathcal{R})$, of polyderivations of an Abelian algebra $\mathcal{R}$. The algebra $(\mathcal{P}(\mathcal{R}), \cdot, [-.-]_S)$ is a G-algebra: the dot product is induced from that on $\mathcal{R}$ using[2] $(a\cdot b)(x) \;=\; a\cdot b(x) + (-1)^{|b|}a(x)\cdot b$; and the bracket operation is the natural generalization of the Schouten-Nijenhuis bracket [22,23] on polyvectors.

We may also construct a large class of natural G-modules in this context. If M is a module of $\mathcal{R}$ on which the Lie algebra $\mathcal{D}(\mathcal{R})$ acts by derivations of the dot product action of $\mathcal{R}$, then the space of polyderivations $\mathcal{P}(\mathcal{R},M)$ naturally has the structure of a G-module of $(\mathcal{P}(\mathcal{R}), \cdot, [-,-]_S)$. The dot and bracket operations are induced rather similarly to the construction of $(\mathcal{P}(\mathcal{R}), \cdot, [-,-]_S)$ above.

A.5. Explicit examples

The simplest example of the construction of a G-algebra of polyderivations $\mathcal{P}(\mathcal{R})$ as discussed above, is to take $\mathcal{R}$ to be a freely generated algebra. Let $\mathcal{C}_N \cong \mathbb{C}\,[x^1,\ldots,x^N]$ be a free Abelian algebra on N generators. It is straightforward to verify that the G-algebra, $\mathcal{P}(\mathcal{C}_N)$, is nothing but the algebra of polynomial polyvector fields on $\mathbb{C}^N$, *i.e.*,

$$\mathcal{P}(\mathcal{C}_N) \;\cong\; \bigoplus_{n=0}^{N} \bigwedge\nolimits^n \mathcal{D}(\mathcal{C}_N)\,. \tag{A.6}$$

More explicitly, it is a free $\mathbb{Z}_2$-graded algebra with even generators $x^1,\ldots,x^N\in\mathcal{C}_N$ and odd generators $x_1^*,\ldots,x_N^*\in\mathcal{D}(\mathcal{C}_N)$, where $x_i^*(x^j)=\delta_i^j$, $i,j=1,\ldots,N$.

The operator $\Delta_S = -\frac{\partial}{\partial x^i}\frac{\partial}{\partial x_i^*}$ is a second order derivation on $\mathcal{P}(\mathcal{C}_N)$. By direct calculation one finds that the bracket induced by Δ_S is equal to the Schouten-Nijenhuis bracket; so $(\mathcal{P}(\mathcal{C}_N), \cdot, \Delta_S)$ is a BV-algebra. Note that there is a canonical polyvector of maximal order, the “volume element,” $\Omega = \frac{1}{N!}\epsilon^{i_1\cdots i_N}x_{i_1}^*\ldots x_{i_N}^*$, and that $(\mathcal{P}(\mathcal{C}_N), \cdot, \Delta_S)$, as a BV-algebra, is generated by $x^1,\ldots,x^N$ and Ω.

For $N=2$ this is precisely the BV-algebra used in Section 2 to model the BV-algebra of operator cohomology. Let us consider the natural G-modules in this case. We may introduce on $\mathcal{C}_2$ two different ground ring $(\mathcal{C}_2)$ modules: M_1, isomorphic to the ground ring itself, and the twisted module M_r defined by the “dual” realization of the ground ring generators, $x_1\to-\frac{\partial}{\partial x_2}$ and $x_2\to-\frac{\partial}{\partial x_1}$. Equivalently, we may identify M_r as the space freely generated by $\frac{\partial}{\partial x^i}$ on one generator, δ, satisfying $x^i\delta=0$. The ring $\mathcal{C}_2$ now acts simply by multiplication. Similarly, the space $\mathcal{P}(\mathcal{C}_2,M_r)$ is freely generated by $\frac{\partial}{\partial x^i}$ and x_i^* from δ. It is clear from the discussion in Section A.4 that $\mathcal{P}(\mathcal{C}_2,M_r)$ is a G-module. In fact, it is manifestly a BV-module, with the BV-operator realized by $x^2\frac{\partial}{\partial x_1^*}+x^1\frac{\partial}{\partial x_2^*}$.

[2] We define, for simplicity of notation, $a(x)=0$ for $a\in\mathcal{R}$.

A more complicated example is given by the polyderivations of an Abelian algebra which is not free, but whose generators satisfy a single quadratic relation. Consider the Abelian algebra $\mathcal{R}_N = \mathcal{C}_{2N}/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by the vanishing relation

$$h_{ij}\, x^i \cdot x^j \;=\; 0\,, \tag{A.7}$$

and the metric h has nonzero entries $(h_{i(N+j)}) = (h_{(N+i)j}) = \delta_{ij}$.

A polyderivation $\Phi \in \mathcal{P}^n(\mathcal{R}_N)$ is completely determined by its values on the ground ring generators. Therefore, on expanding in the dual basis, $\Phi = \Phi^{i_1\ldots i_n} x^*_{i_1} \ldots x^*_{i_n}$ where $\Phi^{i_1\ldots i_n} \in \mathcal{R}_N$, Φ is a polyderivation iff it preserves the vanishing relation (A.7); *i.e.*, the coefficients of its expansion satisfy

$$x_i \cdot \Phi^{i\, i_1\ldots i_{n-1}} \;=\; 0\,, \quad i_1, \ldots, i_{n-1} = 1, \ldots, 2N\,. \tag{A.8}$$

The free algebra $\mathcal{C}_{2N}$ carries a natural action of the Lie algebra $\mathfrak{so}_{2N}$ realized by the first order derivations $\Lambda_{ij} = x_i x^*_j - x_j x^*_i\,, i,j = 1, \ldots, 2N$. Clearly, $\Lambda_{ij}(\mathcal{I}) \subset \mathcal{I}$, so the action of $\mathfrak{so}_{2N}$ descends to the ground ring $\mathcal{R}_N$, with the generators x^i transforming in the vector ($2N$-dimensional) representation. Using this $\mathfrak{so}_{2N}$ action we can find an explicit basis for the space of polyderivations, $\mathcal{P}(\mathcal{R}_N)$, and a finite set of generators and relations which characterize $\mathcal{P}(\mathcal{R}_N)$ as a dot algebra.

Theorem A.1 [1]. *The graded, graded commutative algebra $(\mathcal{P}(\mathcal{R}_N), \cdot)$ is generated, as a dot algebra, by 1, the ground ring generators x^i, the order one derivation $C = x^i x^*_i$, and the order $n-1$ polyderivations $P_{i_1,i_2\ldots i_n} = x_{[i_1} x^*_{j_1} \ldots x^*_{j_n]}$, $n = 2, \ldots, 2N$, satisfying the relations:*

$$x_i x^i \;=\; 0\,, \tag{A.9}$$

$$x_{[i} P_{i_1,i_2\ldots i_n]} \;=\; 0\,, \tag{A.10}$$

$$x^i P_{i\,,j_1\ldots j_n} \;=\; -\tfrac{n}{n+1} C P_{j_1,j_2\ldots j_n}\,, \tag{A.11}$$

$$P_{i_1,i_2\ldots i_m} P_{j_1,j_2\ldots j_n} \;=\; (-1)^{m-1}\, \tfrac{m+n-1}{n} x_{[i_1} P_{i_2,i_3\ldots i_m]j_1\ldots j_n}\,, \tag{A.12}$$

$$C P_{i_1,i_2\ldots i_{2N}} \;=\; 0\,. \tag{A.13}$$

These results allow us to demonstrate that $\mathcal{P}(\mathcal{R}_N)$ is actually a BV-algebra by construction of the BV-operator Δ_S on this basis, and to explicitly calculate the homology of Δ_S. There is a unique nontrivial homology state, the "volume element" $X = \frac{1}{(2N)!} \epsilon^{i_1 i_2 \ldots i_{2N}}\, x_{i_1} x^*_{i_2} \ldots x^*_{i_{2N}}$.

For comparison with the operator cohomology of the $4D$ $\mathcal{W}_3$ string, the relevant case is $N = 3$. On the one hand, the ring $\mathcal{R}_3$ decomposes under $\mathfrak{sl}_3 \subset \mathfrak{so}_6$ as a direct sum of all irreducible finite-dimensional modules of $\mathfrak{sl}_3$, each module occurring with multiplicity one – *i.e.*, $\mathcal{R}_3$ is a model space for $\mathfrak{sl}_3$. The generators x^i of $\mathcal{R}_3$ decompose as an $\mathfrak{sl}_3$ triplet x_σ and an anti-triplet x^σ and satisfy the vanishing relation $x_\sigma x^\sigma = 0$. On the other hand, $\mathcal{R}_3$ is an irreducible representation of the algebra $\mathfrak{so}_8$ which includes the generators x^i as well as the $\mathfrak{so}_6$ generators Λ_{ij} [24]. The $\mathfrak{so}_8$ generators decompose under $\mathfrak{sl}_3$ as $\mathrm{ad}_{\mathfrak{so}_8} = \mathbf{8} \oplus (\mathbf{3} \oplus \overline{\mathbf{3}}) \oplus (\mathbf{3} \oplus \overline{\mathbf{3}}) \oplus (\mathbf{3} \oplus \overline{\mathbf{3}}) \oplus \mathbf{1} \oplus \mathbf{1}$. We find that there are three ways to extend the $\mathfrak{so}_6$ algebra to $\mathfrak{so}_8$ ($\mathrm{ad}_{\mathfrak{so}_6} = \mathbf{8} \oplus (\mathbf{3} \oplus \overline{\mathbf{3}}) \oplus \mathbf{1}$). For each choice there are two ways to realize the ring generators in terms of the remaining $\mathbf{3}$ and $\bar{\mathbf{3}}$. Thus, there are in total six natural module structures on $\mathcal{R}_3$, which we label M_w, $w \in W(\mathfrak{sl}_3)$.

The isomorphism between these different $\mathfrak{so}_8$ realizations, together with Theorem A.1, are enough to show that for each $w \in W(\mathfrak{sl}_3)$, the space of polyderivations with coefficients in the ground ring module M_w, $\mathcal{P}(\mathcal{R}_3, M_w)$, is a G-module of $\mathcal{P}(\mathcal{R}_3)$. We conjecture that they are, in fact, BV-modules. These modules play an important role in the description of the operator cohomology given in Section 3.

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