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The Generic World-Sheet Action of Irrational Conformal Field Theory * †

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Abstract

We review developments in the world-sheet action formulation of the generic irrational conformal field theory, including the non-linear and the linearized forms of the action. These systems form a large class of spin-two gauged WZW actions which exhibit exotic gravitational couplings. Integrating out the gravitational field, we also speculate on a connection with sigma models.

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1 Affine Lie Algebra and Conformal Field Theory

Affine Lie algebra, or current algebra on the circle, was discovered independently in mathematics [1] and physics [2]. The affine algebras have played a central role in the construction of new conformal field theories [2, 3, 4, 5, 6], which have historically been found first in the Hamiltonian or operator formulation and only later in their corresponding action formulations [7, 8, 9, 10, 11].

One reason for this order of events is that Hamiltonians with different local symmetries correspond to qualitatively different actions, so that generalization to larger classes of actions is not always straightforward. For example, the affine-Sugawara construction [2, 3, 4] is described by the WZW action [7, 8], the coset constructions [2, 3, 5] are described by the spin-one gauged WZW actions [9, 10], and the generic irrational conformal field theory [6, 12] is described by the spin-two gauged WZW actions [11, 13].

On the other hand, all these theories are uniformly included in the general affine-Virasoro construction [6, 14],

$$T = L^{ab} {}^* J_a J_b {}^* \quad , \quad a, b = 1 \dots \dim g \quad (1.1)$$

which is quadratic in the currents J_a of the general affine algebra \hat{g} . The coefficient L^{ab} is called the inverse inertia tensor, in analogy with the spinning top. In the general case, the inverse inertia tensor must satisfy the Virasoro master equation [6, 14], whose solutions include the affine-Sugawara and coset constructions, and a very large number of new conformal field theories.

The generic affine-Virasoro construction has irrational central charge, even on positive integer level of the compact affine algebras. As a consequence, the study of this class of theories is called irrational conformal field theory (ICFT),

$$\text{ICFT} \supset \supset \text{RCFT} \quad (1.2)$$

which includes rational conformal field theory (RCFT) as a small subspace.

The development of ICFT has moved through a number of stages, including:

- exact unitary irrational solutions [12] of the master equation
- partial classification [15, 16] of the solution space of the master equation
- generalized KZ equations on the sphere [17] and the torus [18]
- the generic world-sheet action [11, 13] of ICFT.

See Ref. [19] for a broad review of irrational conformal field theory.

In this talk, we focus on the Hamiltonian and action formulation of the generic ICFT on simple g . In this formulation, the geometry of the action is determined by the local symmetry group of the Hamiltonian. The local symmetry of the generic Hamiltonian is smaller than that of the coset constructions, and the corresponding generic action is a large set of spin-two gauged WZW models. The spin-two gauge fields are gravitational fields with exotic matter couplings which generalize and include the coupling of the conventional world-sheet metric. We will discuss in particular the non-linear and linearized forms of the action and a speculative connection with sigma models.

2 Fundamentals of ICFT

We begin by briefly reviewing some basic facts about ICFT which are essential to understanding the action formulation.

For any Lie g , the general current algebra \hat{g} is [1, 2]

$$J_a(z)J_b(w) = \frac{G_{ab}}{(z-w)^2} + if_{ab}{}^c \frac{J_c(w)}{z-w} + \text{reg.} \quad (2.1)$$

where $a, b = 1 \dots \dim g$, and $f_{ab}{}^c$ and G_{ab} are the structure constants and generalized metric of g . For the example of simple compact g , one has $G_{ab} = k\eta_{ab}$, where η_{ab} is the Killing metric of g and k is the level of the affine algebra.

On each level of \hat{g} , the general affine-Virasoro construction is [6, 14]

$$T(z) = L^{ab} {}_* J_a(z) J_b(z) {}_* \quad (2.2a)$$

$$L^{ab} = 2L^{ac}G_{cd}L^{db} - L^{cd}L^{ef}f_{ce}{}^a f_{df}{}^b - L^{cd}f_{ce}{}^f f_{df}{}^{(a} L^{b)e} \quad (2.2b)$$

$$c = 2G_{ab}L^{ab} \quad (2.2c)$$

where (2.2b) is the Virasoro master equation. The master equation is a large set of coupled quadratic equations for the inverse inertia tensor L^{ab} . For each solution L^{ab} of the master equation, one obtains a conformal field theory with central charge c given in (2.2c). Generically-irrational central charge, even on positive integer levels of the affine algebra, is an immediate consequence of the structure of the master equation. In what follows, we confine our remarks to simple compact g , which is all that is needed below.

a) Affine-Sugawara construction [2, 3, 4]. The affine-Sugawara construction is the simplest solution of the master equation, with

$$L_g^{ab} = \frac{\eta^{ab}}{2k + Q_g} \quad , \quad c_g = \frac{2k \dim g}{2k + Q_g} \quad (2.3)$$

where $Q_g \eta_{ab} = -f_{ac}{}^d f_{bd}{}^c$.

b) K-conjugation covariance [2, 6, 3]. A central feature of the master equation is that its solutions come in K-conjugate pairs L and \tilde{L} , where

$$L^{ab} + \tilde{L}^{ab} = L_g^{ab} \quad , \quad c + \tilde{c} = c_g. \quad (2.4)$$

The corresponding K-conjugate stress tensors T and \tilde{T} ,

$$T(z)\tilde{T}(w) = \text{reg.} \quad , \quad T(z) + \tilde{T}(z) = T_g(z) \quad (2.5)$$

commute and sum to the affine-Sugawara stress tensor T_g .

K-conjugation is used to generate additional solutions, as in the familiar case of the coset constructions [2, 3, 5] $T_h + T_{g/h} = T_g$. At the level of dynamics, K-conjugation also provides the minimal local symmetry of any ICFT, as discussed below.

c) Semi-classical solutions [12, 20]. On simple g , the generic solutions of the master equation live in level-families $L^{ab}(k)$ whose high-level forms are

$$L^{ab} = \frac{P^{ab}}{2k} + \mathcal{O}(k^{-2}) \quad , \quad \tilde{L}^{ab} = \frac{\tilde{P}^{ab}}{2k} + \mathcal{O}(k^{-2}) \quad (2.6a)$$

$$L^{ab} + \tilde{L}^{ab} = L_g^{ab} = \frac{\eta^{ab}}{2k} + \mathcal{O}(k^{-2}) \quad (2.6b)$$

$$c = \text{rank } P + \mathcal{O}(k^{-1}) \quad , \quad \tilde{c} = \text{rank } \tilde{P} + \mathcal{O}(k^{-1}) \quad , \quad c_g = \dim g + \mathcal{O}(k^{-1}) \quad (2.6c)$$

$$P^2 = P \quad , \quad \tilde{P}^2 = \tilde{P} \quad , \quad P\tilde{P} = 0 \quad , \quad P + \tilde{P} = 1 \quad (2.6d)$$

where P and \tilde{P} are the high-level projectors of the L and the \tilde{L} theory respectively.

The partial classification [15, 16] of ICFT by graph theory is based on these high-level forms, and it is believed [19] that the generic level-family is generically unitary on positive integer levels of \hat{g} .

d) Spin-two gauge theories [11, 21, 13]. The basic Hamiltonian of the general affine-Virasoro construction is

$$H_0 = L_0 + \bar{L}_0 = L^{ab} ({}^* J_a J_b + \bar{J}_a \bar{J}_b {}^*)_0 \quad (2.7)$$

where the barred currents \bar{J} are right-mover copies of the left-mover currents J . For the g/h coset constructions the symmetry algebra of H_0 is affine $h \times h$, which leads to a world-sheet description by spin-one (or Lie-algebra) gauged WZW models [9, 10].

In the space of all conformal field theories, the coset constructions are only special points of higher symmetry. For the generic affine-Virasoro construction L^{ab} , the symmetry group of H_0 is $\text{Diff } S_1 \times \text{Diff } S_1$, generated by the left- and right-mover stress tensors

$$\tilde{T} = \tilde{L}^{ab} {}^* J_a J_b {}^* \quad , \quad \bar{\tilde{T}} = \tilde{L}^{ab} {}^* \bar{J}_a \bar{J}_b {}^* \quad (2.8)$$

of the K-conjugate theory \tilde{L}^{ab} . As a consequence, the world-sheet action of the generic ICFT is a spin-two gauged WZW model, as discussed in the following section.

3 The Exotic Gravities of ICFT

3.1 The Generic Affine-Virasoro Hamiltonian

The classical basic Hamiltonian of the generic level-family $L^{ab}(k)$ on simple g is

$$H_0 = \int_0^{2\pi} d\sigma \mathcal{H}_0 \quad , \quad \mathcal{H}_0 = \frac{1}{2\pi} L_\infty^{ab} (J_a J_b + \bar{J}_a \bar{J}_b) \quad (3.1)$$

where $L_\infty^{ab} = P^{ab}/2k$ is the high-level form of L^{ab} in Eq. (2.6). The classical currents J_a, \bar{J}_a are taken as Bowcock's canonical forms [22], which satisfy the bracket algebra of affine $g \times g$. The coset constructions, with (bracket) affine symmetry, are included in (3.1) when $L = L_{g/h} = L_g - L_h$, but we consider only the generic L^{ab} for which, as in the quantum theory, the local symmetry algebra of H_0 is $\text{Diff } S_1 \times \text{Diff } S_1$. The generators of the diffeomorphism groups are the conformal stress tensors of the commuting K-conjugate theory,

$$\tilde{L}_\infty^{ab} J_a J_b \quad , \quad \bar{\tilde{L}}_\infty^{ab} \bar{J}_a \bar{J}_b \quad (3.2)$$

where $\tilde{L}_\infty^{ab} = \tilde{P}^{ab}/2k$ is the high-level form of \tilde{L}^{ab} in Eq. (2.6). Since these classical generators satisfy the Virasoro algebra without central extension, they form a set of first class constraints of H_0 .

Following Dirac, one obtains the full Hamiltonian [11] of the generic theory L ,

$$H = \int_0^{2\pi} d\sigma \mathcal{H} \quad , \quad \mathcal{H} = \mathcal{H}_0 + v \cdot K(\tilde{L}_\infty) \quad (3.3a)$$

$$v \cdot K(\tilde{L}_\infty) = \frac{1}{2\pi} \tilde{L}_\infty^{ab} (v J_a J_b + \bar{v} \bar{J}_a \bar{J}_b) \quad (3.3b)$$

where the K -conjugate stress tensors in $v \cdot K$ play the role of Gauss' law and v, \bar{v} are multipliers. The multipliers form a spin-two gauge field or gravitational field on the world-sheet, which is called the K -conjugate metric. This Hamiltonian generalizes and includes the WZW Hamiltonian (which is included when $P = 1, \tilde{P} = 0$) and the conventional world-sheet metric formulation of the WZW model (which is included when $P = 0, \tilde{P} = 1$).

More generally, the K -conjugate metric is an exotic gravity because it exhibits exotic, \tilde{L}^{ab} -dependent coupling only to the “ K -conjugate matter”, which is, loosely speaking, only “half” the matter.

3.2 The Non-Linear Action

The action corresponding to H in (3.3) is the non-linear form of the generic affine-Virasoro action [11],

$$S = \int d\tau d\sigma (\mathcal{L} + \Gamma) \quad (3.4a)$$

$$\begin{aligned} \mathcal{L} = & \frac{1}{8\pi} e_i^a G_{bc} \epsilon_j^c \left[\left[f(Z) + \alpha \bar{\alpha} \omega \tilde{P} \omega^{-1} f(Z) \tilde{P} \right]_a^b \left(\dot{x}^i \dot{x}^j - x^i x'^j \right) \right. \\ & + \alpha \left[f(Z) \tilde{P} \right]_a^b \left(\dot{x}^i \dot{x}^j + x^i x'^j + \dot{x}^{(i} x^{j)'} \right) \\ & + \bar{\alpha} \left[\omega \tilde{P} \omega^{-1} f(Z) \right]_a^b \left(\dot{x}^i \dot{x}^j + x^i x'^j - \dot{x}^{(i} x^{j)'} \right) \\ & \left. + \left[1 - f(Z) + \alpha \bar{\alpha} \omega \tilde{P} \omega^{-1} f(Z) \tilde{P} \right]_a^b \left(\dot{x}^{[i} x^{j]'} \right) \right] \end{aligned} \quad (3.4b)$$

$$f(Z) \equiv [1 - \alpha \bar{\alpha} Z]^{-1}, \quad Z \equiv \tilde{P} \omega \tilde{P} \omega^{-1}, \quad \alpha \equiv \frac{1-v}{1+v}, \quad \bar{\alpha} \equiv \frac{1-\bar{v}}{1+\bar{v}} \quad (3.4c)$$

which is the world-sheet description of the generic theory L . Here, x^i and $e_i^a, i = 1 \dots \dim g$ are the coordinates and left-invariant vielbein on the group manifold G . Also, $\omega(g)_a{}^b$ is the adjoint action of $g \in G$, and Γ is the WZW term. The non-linear action reduces to the WZW action when $\tilde{P} = 0$, and to the world-sheet metric formulation of the WZW model when $\tilde{P} = 1$.

We call attention to the exotic, non-linear coupling of the K-conjugate metric $(\alpha, \bar{\alpha})$ in the action (3.4). In spite of this non-linearity, the action exhibits Lorentz, conformal and local Weyl symmetries, and the expected world-sheet diffeomorphism invariance. The diffeomorphism group is called $\text{Diff } S_2(K)$ because it is associated to the commuting K-conjugate theory.

The K-conjugate metric can be written in standard form,

$$\tilde{h}_{mn} \equiv e^{-\phi} \begin{pmatrix} -v\bar{v} & \frac{1}{2}(v - \bar{v}) \\ \frac{1}{2}(v - \bar{v}) & 1 \end{pmatrix}, \quad \sqrt{-\tilde{h}}\tilde{h}^{mn} = \frac{2}{v + \bar{v}} \begin{pmatrix} -1 & \frac{1}{2}(v - \bar{v}) \\ \frac{1}{2}(v - \bar{v}) & v\bar{v} \end{pmatrix} \quad (3.5)$$

and \tilde{h}_{mn} transforms under $\text{Diff } S_2(K)$ as a second-rank tensor field. The $\text{Diff } S_2(K)$ transformations of the matter are given in Refs. [11] and [13]. In the discussion below, we give the corresponding matter transformations for the linearized form of the action.

The gravitational stress tensor of the K-conjugate metric is defined in the usual way,

$$\tilde{\theta}^{mn} = \frac{2}{\sqrt{-\tilde{h}}} \frac{\delta S}{\delta \tilde{h}_{mn}} \quad (3.6)$$

and, in the conformal gauge,

$$v = \bar{v} = 1 \quad , \quad \alpha = \bar{\alpha} = 0 \quad , \quad \sqrt{-\tilde{h}}\tilde{h}^{mn} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.7)$$

this prescription reproduces the conformal stress tensor of the \tilde{L} theory, as it should. It follows that \tilde{h}_{mn} is the world-sheet metric of the \tilde{L} theory.

The chiral currents of the underlying affine-Virasoro construction (2.2a) are also found [11] in the conformal gauge. In other gauges, the currents are gauge-equivalent to chiral currents.

3.3 The Linearized Action

In the equivalent linearized form [13], the generic action is clearly seen as a large class of spin-two gauged WZW models,

$$S' = S_{WZW} + \int d^2z \Delta \mathcal{L}_B \quad (3.8a)$$

$$\begin{aligned} \Delta \mathcal{L}_B &= \frac{\alpha}{\pi y^2} \tilde{L}_\infty^{ab} \text{Tr}(\mathcal{T}_a B) \text{Tr}(\mathcal{T}_b B) \\ &\quad + \frac{\bar{\alpha}}{\pi y^2} \tilde{L}_\infty^{ab} \text{Tr}(\mathcal{T}_a \bar{B}) \text{Tr}(\mathcal{T}_b \bar{B}) \\ &\quad - \frac{1}{\pi y} \text{Tr}(\bar{D}_B g D_B g^{-1}) \end{aligned} \quad (3.8b)$$

$$D_B \equiv \partial + iB \quad , \quad \bar{D}_B \equiv \bar{\partial} + i\bar{B}. \quad (3.8c)$$

Here, $g(\mathcal{T})$ is the group element in irrep \mathcal{T} of g , S_{WZW} is the Wess-Zumino-Witten action and $y \sim k^{-1}$ is associated to the trace normalization of \mathcal{T} . The quantities $B = B^a \mathcal{T}_a$, $\bar{B} = \bar{B}^a \mathcal{T}_a$ are a set of auxiliary fields, called the connections for reasons which will be clear below. Integration of the connections gives the non-linear form (3.4) of the action.

The action (3.8) describes the generic theory L as a spin-two gauging of the WZW action by the K-conjugate theory \tilde{L} . The couplings of the gauge field $\tilde{h}_{mn}(\alpha, \bar{\alpha})$ are quite simple in this form. The linearized action is invariant under the Diff $S_2(K)$ transformations

$$\delta \alpha = -\bar{\partial} \xi + \xi \overleftrightarrow{\partial} \alpha, \quad \delta \bar{\alpha} = -\partial \bar{\xi} + \bar{\xi} \overleftrightarrow{\partial} \bar{\alpha} \quad (3.9a)$$

$$\delta g = g i \lambda - i \bar{\lambda} g \quad (3.9b)$$

$$\delta B = \partial \lambda + i[B, \lambda], \quad \delta \bar{B} = \bar{\partial} \bar{\lambda} + i[\bar{B}, \bar{\lambda}] \quad (3.9c)$$

$$\lambda \equiv \lambda^a \mathcal{T}_a, \quad \bar{\lambda} \equiv \bar{\lambda}^a \mathcal{T}_a \quad (3.9d)$$

$$\lambda^a \equiv 2\xi \tilde{L}_\infty^{ab} B_b, \quad \bar{\lambda}^a \equiv 2\bar{\xi} \tilde{L}_\infty^{ab} \bar{B}_b \quad (3.9e)$$

where $\xi, \bar{\xi}$ are the diffeomorphism parameters. The transformation of the K-conjugate metric in (3.9a) is the usual transformation of a second-rank tensor

field, as noted above, but the Diff $S_2(K)$ transformation of B, \bar{B} and the matter is quite exotic.

In particular, Eqs. (3.9b,c) show that Diff $S_2(K)$ is locally embedded in local Lie $g \times \text{Lie } g$, with the matter $g(\mathcal{T}) \in G$ and the connections B, \bar{B} transforming under local Lie $g \times \text{Lie } g$ as the group element and the Lie $g \times \text{Lie } g$ connection respectively. These transformation properties account for the covariant derivatives in (3.8c) and the intriguing resemblance of the linearized action (3.8) to the usual (Lie algebra) gauged WZW model. Further discussion of the local embedding and the linearized action is given in Refs. [11], [13], and [19].

4 Two world-sheet metrics

We have seen that the world-sheet action of the generic theory L is a spin-two gauge theory, where the gauge field \tilde{h}_{mn} is the world-sheet metric of the \tilde{L} theory. Because \tilde{h}_{mn} couples only to the \tilde{L} matter, it is also possible [11] to introduce another spin-two gauge field h_{mn} ,

$$h_{mn} \equiv e^{-\chi} \begin{pmatrix} -u\bar{u} & \frac{1}{2}(u - \bar{u}) \\ \frac{1}{2}(u - \bar{u}) & 1 \end{pmatrix}, \quad \sqrt{-h}h^{mn} = \frac{2}{u + \bar{u}} \begin{pmatrix} -1 & \frac{1}{2}(u - \bar{u}) \\ \frac{1}{2}(u - \bar{u}) & u\bar{u} \end{pmatrix} \quad (4.1)$$

which is the world-sheet metric of the L theory. This results in the *doubly-gauged action* S_D , with a K-conjugate pair of world-sheet metrics \tilde{h}_{mn} and h_{mn} .

The linearized form of the doubly-gauged action [13] is surprisingly simple,

$$S_D = S_{WZW} + \int d^2z \Delta\mathcal{L}_D \quad (4.2a)$$

$$\begin{aligned} \Delta\mathcal{L}_D &= \frac{1}{\pi y^2} (\alpha \tilde{L}_\infty^{ab} + \beta L_\infty^{ab}) \text{Tr}(\mathcal{T}_a B) \text{Tr}(\mathcal{T}_b B) \\ &\quad + \frac{1}{\pi y^2} (\bar{\alpha} \tilde{L}_\infty^{ab} + \bar{\beta} L_\infty^{ab}) \text{Tr}(\mathcal{T}_a \bar{B}) \text{Tr}(\mathcal{T}_b \bar{B}) \\ &\quad - \frac{1}{\pi y} \text{Tr}(\bar{D}_B g D_B g^{-1}) \end{aligned} \quad (4.2b)$$

$$\alpha = \frac{1-v}{1+v}, \quad \bar{\alpha} = \frac{1-\bar{v}}{1-\bar{v}}, \quad \beta = \frac{1-u}{1+u}, \quad \bar{\beta} = \frac{1-\bar{u}}{1+\bar{u}} \quad (4.2c)$$

and may in fact be obtained from the action (3.8) by the substitution

$$\alpha \tilde{L}_\infty^{ab} \rightarrow \alpha \tilde{L}_\infty^{ab} + \beta L_\infty^{ab}, \quad \bar{\alpha} \tilde{L}_\infty^{ab} \rightarrow \bar{\alpha} \tilde{L}_\infty^{ab} + \bar{\beta} L_\infty^{ab}. \quad (4.3)$$

Other forms of the doubly-gauged action, including the non-linear form, are given in Refs. [13] and [19].

The doubly-gauged action (4.2) is invariant under two commuting diffeomorphism groups [13], called $\text{Diff } S_2(T) \times \text{Diff } S_2(K)$, which are associated to the two world-sheet metrics h_{mn} and \tilde{h}_{mn} . In particular, h_{mn} is a rank-two tensor under $\text{Diff } S_2(T)$ but inert under $\text{Diff } S_2(K)$ and vice versa for \tilde{h}_{mn} . The $\text{Diff } S_2(T) \times \text{Diff } S_2(K)$ transformations of the matter and the connections are given in Ref. [13]

In the doubly-gauged action (4.2), the K-conjugate metrics $\tilde{h}_{mn}(\alpha, \bar{\alpha})$ and $h_{mn}(\beta, \bar{\beta})$ are on equal footing. When considering the L theory, however, only \tilde{h}_{mn} is dynamical, while h_{mn} provides a gravitational probe for the stress tensor of the L theory,

$$\theta^{mn} = \frac{2}{\sqrt{-h}} \frac{\delta S_D}{\delta h_{mn}} \quad (4.4)$$

in parallel with the stress tensor $\tilde{\theta}^{mn}$ of the \tilde{L} theory in (3.6). In what follows, we refer to \tilde{h}_{mn} and h_{mn} as the \tilde{L} -metric and the L -metric respectively.

5 A Connection with Sigma Models

An important open problem in the action formulation of ICFT is the possible connection with sigma models. In this section we sketch a speculative, essentially classical derivation [19] of such a connection, with surprising results. The details of this derivation cannot be taken seriously until one-loop effects are properly included.

One begins with the doubly-gauged action (4.2) for the L theory, and integrates the dynamical gauge fields $\alpha, \bar{\alpha}$ of the \tilde{L} metric. This gives the δ -function constraints on the connections,

$$\tilde{P}^{ab} B_a B_b = \tilde{P}^{ab} \bar{B}_a \bar{B}_b = 0 \quad (5.1)$$

which are solved by $B_a = P_a{}^b b_b$, $\bar{B}_a = P_a{}^b \bar{b}_b$ with unconstrained b, \bar{b} . Then, one may integrate b, \bar{b} to obtain the action

$$S_{\beta, \bar{\beta}} = \int d\tau d\sigma (\mathcal{L} + \Gamma) \quad (5.2a)$$

$$\mathcal{L} = -\frac{1}{8\pi} G_{ab} (e_\tau{}^a e_\tau{}^b - e_\sigma{}^a e_\sigma{}^b) - \frac{1}{8\pi} E^A (C^{-1})_A{}^B E_B \quad (5.2b)$$

$$E^A = ((e_\tau{}^a - e_\sigma{}^a), (\bar{e}_\tau{}^a + \bar{e}_\sigma{}^a)) \quad , \quad E_B = \begin{pmatrix} G_{bc} (e_\tau{}^c - e_\sigma{}^c) \\ G_{bc} (\bar{e}_\tau{}^c + \bar{e}_\sigma{}^c) \end{pmatrix} \quad (5.2c)$$

$$C^{-1} = \begin{pmatrix} -\bar{\beta} P M(\omega) P & P \omega P M(\omega^{-1}) P \\ P \omega^{-1} P M(\omega) P & \beta P M(\omega^{-1}) P \end{pmatrix} \quad (5.2d)$$

$$M(\omega) \equiv ((1 + P(\omega - 1)P)(1 + P(\omega^{-1} - 1)P) - \beta \bar{\beta} P)^{-1} \quad (5.2e)$$

which is the conformal field theory of L coupled to its world-sheet metric $h_{mn}(\beta, \bar{\beta})$. Here, $e_\tau{}^a = e_i{}^a \dot{x}^i$, $e_\sigma{}^a = e_i{}^a x'^i$, and the barred quantities are defined similarly with the right-invariant vielbeins $\bar{e}_i{}^a$.

Using the action (5.2) and the prescription (4.4), one computes the stress tensor of the L theory,

$$\begin{aligned} \theta_{00} = \theta_{11} &= \frac{1}{16\pi} (e_\tau{}^a - e_\sigma{}^a)(e_\tau{}^b - e_\sigma{}^b) G_{bc} (PF(\omega)P)_a{}^c \\ &+ \frac{1}{16\pi} (\bar{e}_\tau{}^a + \bar{e}_\sigma{}^a)(\bar{e}_\tau{}^b + \bar{e}_\sigma{}^b) G_{bc} (PF(\omega^{-1})P)_a{}^c \\ \theta_{01} = \theta_{10} &= \frac{1}{16\pi} (e_\tau{}^a - e_\sigma{}^a)(e_\tau{}^b - e_\sigma{}^b) G_{bc} (PF(\omega)P)_a{}^c \\ &- \frac{1}{16\pi} (\bar{e}_\tau{}^a + \bar{e}_\sigma{}^a)(\bar{e}_\tau{}^b + \bar{e}_\sigma{}^b) G_{bc} (PF(\omega^{-1})P)_a{}^c \end{aligned} \quad (5.3a)$$

$$F(\omega) \equiv ((1 + P(\omega - 1)P)(1 + P(\omega^{-1} - 1)P))^{-1} \quad (5.3b)$$

in the conformal gauge ($\beta = \bar{\beta} = 0$) of the L metric. The matter degrees of freedom in (5.3) are entirely projected onto the P subspace. As a consequence, one finds that the stress tensor is conformal at high level with high-level central charge

$$c(L) = \text{rank } P + \mathcal{O}(k^{-1}) \quad (5.4)$$

as it should be for the L theory (see Eq. (2.6c)).

Finally, one obtains the conformal field theory of L as the effective sigma model,

$$S_{\text{eff}} = \int d\tau d\sigma \left[\frac{1}{8\pi} G_{ij}^{\text{eff}} (\dot{x}^i \dot{x}^j - x'^i x'^j) + \frac{1}{4\pi y} B_{ij}^{\text{eff}} \dot{x}^i x'^j \right] \quad (5.5a)$$

$$G_{ij}^{\text{eff}} = e_i^a e_j^b G_{cb} (N + N^T - 1)_a^c \quad , \quad B_{ij}^{\text{eff}} = B_{ij} - y e_i^a e_j^b G_{cb} (N - N^T)_a^c \quad (5.5b)$$

$$N = \omega P (1 + P(\omega - 1)P)^{-1} P \quad , \quad N^T = P (1 + P(\omega^{-1} - 1)P)^{-1} P \omega^{-1} \quad (5.5c)$$

by evaluating the action (5.2) in the conformal gauge of the L metric. Here B_{ij} is the WZW antisymmetric tensor field, and $G_{ij}^{\text{eff}}, B_{ij}^{\text{eff}}$ are the space-time metric and antisymmetric tensor field on the target space. This sigma model reduces to the WZW model when $P = 1$, as it should.

The result (5.5) is an ordinary sigma model, but the derivation above illustrates the remarkable fact that the question of conformal invariance of a sigma model depends on the choice of the world-sheet metric (the gravitational coupling) and its associated stress tensor.

The correct stress tensor (5.3) of the L theory followed from the exotic coupling of the L metric to the L matter in the action (5.2), but one may also consider the same sigma model (5.5) with the distinct, conventional gravitational coupling [23],

$$\dot{x}^i \dot{x}^j - x'^i x'^j \rightarrow \sqrt{-g_C} g_C^{mn} \partial_m x^i \partial_n x^j \quad (5.6)$$

of the conventional world-sheet metric g_{mn}^C , which gives the conventional stress tensor θ_{mn}^C ,

$$\theta_{mn}^C = \frac{2}{\sqrt{-g_C}} \frac{\delta S}{\delta g_{mn}^C}. \quad (5.7)$$

It is unlikely that θ_{mn}^C is conformal in this case, but the question is not directly relevant because, as we have seen, g_{mn}^C and θ_{mn}^C are not the world-sheet metric and stress tensor of the L theory.

We finally mention that Tseytlin [24] and Bardakçi [25] have studied a similar sigma model, which is related to the bosonization of a generalized Thirring model [26]. When Q in Tseytlin's (3.1) is taken as twice the high- k projector P , his action and the sigma model (5.5) are the same except for a dilaton term and an overall minus sign for the kinetic term. Further investigation will determine whether this intriguing circumstance is more than a coincidence.

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