

INVARIANT EFFECTIVE ACTIONS AND COHOMOLOGY ***Eric D'Hoker***Physics Department**University of California, Los Angeles**Los Angeles, CA 90024, USA*

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Abstract

We review the correspondence between effective actions resulting from non-invariant Lagrangian densities, for Goldstone bosons arising from spontaneous breakdown of a symmetry group G to a subgroup H , and non-trivial generators of the de Rham cohomology of G/H . We summarize the construction of cohomology generators in terms of symmetric tensors with certain invariance and vanishing properties with respect to G and H . The resulting actions in four dimensions arise either from products of generators of lower degree such as the Goldstone-Wilczek current, or are of the Wess-Zumino-Witten type. Actions in three dimensions arise as Chern-Simons terms evaluated on composite gauge fields and may induce fractional spin on solitons.

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1. Introduction

We review recent investigations ^{1,2,3} into the structure of effective actions for Goldstone bosons that arise in the process of spontaneous breakdown of a continuous internal symmetry group G to a subgroup H . The universal nature of spontaneous symmetry breakdown strongly constrains the form of the effective action of the Goldstone fields. The low energy dynamics is completely determined by the groups G and H , and by only a finite number of coupling constants, up to any given order in an expansion in powers of derivatives or momenta.

Long ago, a general method was developed ⁴ for constructing the most general invariant Lagrangian densities for the Goldstone fields. Invariance of the action does not however, in general, require invariance of the Lagrangian density, which may change instead by a total derivative term. The Wess-Zumino-Witten (WZW) term, which was originally considered as an effective action for chiral anomalies, is an example of such an exception ^{5,6}.

The success of the effective field theory approach depends critically on the ability to enumerate all invariant terms in the effective action with a given number of derivatives. Omission of certain terms would lead to an inconsistent correspondence with the dynamics of the underlying microscopic theory and would invalidate the effective action approach. Thus, we are led to investigate the structure of the most general actions, including those that do not arise from G -invariant Lagrangian densities. We shall leave the dimension of space-time arbitrary.

In Ref. 1, we characterized effective Lagrangian densities that, although not G -invariant, yield G -invariant effective actions, in terms of the de Rham cohomology of the coset space G/H . The corresponding invariant effective actions $S[\pi]$ for the Goldstone fields $\pi(x)$ in space-time dimension $n - 1$ are given in terms of the cohomology generators Ω of degree n by

$$S[\pi] = \int_{B_n} \Omega(\tilde{\pi}) \quad (1.1)$$

Here, $n - 1$ dimensional space-time M_{n-1} is extended to a ball B_n with boundary $\partial B_n = M_{n-1}$, and the field $\tilde{\pi}$ interpolates continuously between the original field π on M_{n-1} and the 0 field. Each non-trivial cohomology generator produces a G -invariant action given by (1.1) which arises from a non-invariant Lagrangian density and escapes the construction of G -invariant actions given in Ref 4. The derivation of this correspondence will be briefly reviewed in §2.

The de Rham cohomology is well-known in the case where G/H is itself a Lie group⁷ and the results will be listed in §3.1. For general coset spaces G/H , the structure of de Rham cohomology has been the subject of intense study in mathematics⁸. Yet, the number and the form of the generators does not seem to be available explicitly for general G/H . In Ref. [2], we obtained a simple construction of all de Rham cohomology generators for arbitrary compact G and subgroup H in terms of symmetric tensors with certain invariance and vanishing properties under G and H . This construction will be summarized in §3.2 and §3.3.

The invariant actions produced by this construction have direct physical interpretations. For cohomology of degree 2, they describe the dynamics of charged particles in a magnetic monopole field⁶; for degree 3 they correspond to the WZW term in 2 dimensions. For degree 4, they may be recast in terms of the Chern-Simons invariant^{9,10} evaluated on an \mathcal{H} -valued gauge field built out of the Goldstone fields and they provide generalizations of the Hopf invariant¹¹ to arbitrary G/H . For degree 5, they are WZW terms on cosets G/H , which arise only when there exists a non-vanishing G -invariant symmetric tensor of rank 3 (the d -symbols of chiral anomalies¹⁰) which vanishes upon restriction to the Lie subalgebra of H . The fact that the WZW term can be constructed this way has been known for some time^{12,13}. What was shown in Refs. 1 and 2 is that this construction produces *all Lagrangian densities* that, although not G -invariant, yield G -invariant actions. The arguments are reviewed in §4. Coupling constant quantization in arbitrary dimension and fractionalization of soliton spin in 2+1 dimensions are discussed briefly in §5.

2. Characterizing Invariant Actions by Cohomology of G/H

Let G be a compact connected Lie group and H a subgroup, with Lie algebras \mathcal{G} and \mathcal{H} respectively. \mathcal{G} may be decomposed as $\mathcal{G} = \mathcal{H} + \mathcal{M}$, with $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$, and $[\mathcal{H}, \mathcal{M}] \subset \mathcal{M}$. The Goldstone fields $\pi^a(x)$ transform under linear representations of H , but under non-linear realizations of G . They parametrize the coset G/H in terms of the field $U(\pi)$ in G which, under a global transformation $g \in G$, maps $\pi \rightarrow \pi'$, with

$$gU(\pi) = U(\pi')h(\pi, g) \tag{2.1}$$

* We shall denote indices for the generators of \mathcal{G} , \mathcal{H} and \mathcal{M} by capital $A = 1, \dots, \dim G$, lower case Greek α and lower case a respectively.

where $h(\pi, g)$ is some element of the unbroken subgroup H . The transformation properties of the Lie algebra valued derivatives follow from Eq. (2.1)

$$U^{-1}(\pi') \frac{\partial U(\pi')}{\partial \pi^a} = h(\pi, g) U^{-1}(\pi) \frac{\partial U(\pi)}{\partial \pi^a} h^{-1}(\pi, g) - \frac{\partial h(\pi, g)}{\partial \pi^a} h^{-1}(\pi, g) \quad (2.2)$$

The \mathcal{H} component of Eq. (2.2) transforms as an \mathcal{H} -valued gauge field, while the \mathcal{M} component transforms homogeneously. This allows us to introduce a composite gauge field $V = V_\mu dx^\mu$, and an \mathcal{H} covariant derivative $D^{\mathcal{H}} = D_\mu^{\mathcal{H}} dx^\mu$ as follows

$$V = (U^{-1}dU)_{\mathcal{H}} \quad U^{-1}D^{\mathcal{H}}U = (U^{-1}dU)_{\mathcal{M}} = U^{-1}(dU - UV) \quad (2.3)$$

The most general local invariant Lagrangian density is obtained as a sum of monomials each of which is an invariant product of covariant derivatives [4]

$$\mathcal{L} = \mathcal{L}(U^{-1}D_\mu^{\mathcal{H}}U, \dots, U^{-1}D_{\mu_1}^{\mathcal{H}} \dots D_{\mu_n}^{\mathcal{H}}U, \dots) \quad (2.4)$$

The field strength $W = dV + V^2$ of the gauge field V does not have to enter Eq. (2.4), since it is expressible in terms of the commutator of two $D^{\mathcal{H}}$'s.

We now consider an action $S[\pi]$ which is invariant under G , so that $S[\pi'] = S[\pi]$, but which is not necessarily obtained from an invariant Lagrangian density. Still, we show that its variation with respect to π is an invariant density. First, the variation under an arbitrary change in π may always be written as

$$\delta S[\pi] = \int_{M_{n-1}} d^{n-1}x \operatorname{tr} \{ (U^{-1}\delta U)_{\mathcal{M}} J \} \quad (2.5)$$

where J is a local \mathcal{M} -valued function of π and its derivatives. Since $S[\pi] = S[\pi']$ for all π , the variational derivatives with respect to π are also equal for all π

$$\operatorname{tr} \left\{ \left[U^{-1}(\pi') \frac{\partial U(\pi')}{\partial \pi^a} \right]_{\mathcal{M}} J(\pi') \right\} = \operatorname{tr} \left\{ \left[U^{-1}(\pi) \frac{\partial U(\pi)}{\partial \pi^a} \right]_{\mathcal{M}} J(\pi) \right\} \quad (2.6)$$

In view of Eq. (2.2), the \mathcal{M} component of $U^{-1}\partial U/\partial \pi^a$ transforms homogeneously under G by conjugation with $h(\pi, g)$. Thus J must also transform homogeneously in order to satisfy Eq. (2.6). The \mathcal{M} component of $U^{-1}\delta U$ in Eq. (2.5) transforms homogeneously, again in view of Eq. (2.2), so that the variation of the Lagrangian density $\operatorname{tr}\{(U^{-1}\delta U)_{\mathcal{M}}J\}$ is invariant, as promised. It follows that the contribution from any term in the invariant effective action to the classical equations of motion will be manifestly covariant under G .

The above result leads to a natural n -dimensional formulation of the invariant action. As described in §1, $n - 1$ dimensional space-time M_{n-1} is extended to an n dimensional manifold B_n , of which M_{n-1} is the boundary. We introduce a smooth function $\tilde{\pi}^a(x, t^1)$, which interpolates between $\tilde{\pi}^a(x, 1) = \pi^a(x)$, and $\tilde{\pi}^a(x, 0) = 0$. (Often at this point, space-time is compactified to a sphere S^{n-1} , so that B_n is an n -ball. Here however, we shall not specify any particular topology for M_{n-1} , as there may be physically interesting situations for topologies other than spherical. Our arguments apply in the case of general topology as long as the interpolation $\tilde{\pi}$ exists ^{1.}) The action may then be written in n dimensions

$$S[\pi] = \int_{B_n} dx^{n-1} dt^1 \mathcal{L}_1 \quad (2.7)$$

where \mathcal{L}_1 is the G -invariant density $\text{tr} \{(U^{-1} \partial U / \partial t_1)_{\mathcal{M}} J\}$.

The most general expression of this type can be obtained as in Eq. (2.4), except for the fact that the density \mathcal{L}_1 must satisfy the integrability conditions that guarantee that it arises from a variation of an action. Consider a general deformation $\pi(x) \rightarrow \tilde{\pi}(x; t^i)$, where t^i are a set of $\dim G/H - (n - 1)$ free parameters, that along with the x^μ provide a set of coordinates for G/H . We have shown that

$$\frac{\partial S[\tilde{\pi}]}{\partial t_i} = \int_{M_{n-1}} d^{n-1}x \mathcal{L}_i \quad (2.8)$$

where \mathcal{L}_i are G -invariant functions of $\tilde{\pi}^a$ and its derivatives. Integrability of this system requires that

$$\frac{\partial \mathcal{L}_i}{\partial t^j} - \frac{\partial \mathcal{L}_j}{\partial t^i} = -\partial_\mu \mathcal{L}_{ij}^\mu \quad (2.9)$$

These equations have further integrability conditions on \mathcal{L}_{ij}^μ . It was shown in Ref. 1 that the entire sequence of integrability conditions may be expressed as the closure of a differential form $\Omega(\tilde{\pi})$ of degree n with respect to $d = dt^i \partial_i + dx^\mu \partial_\mu$

$$d\Omega(\tilde{\pi}) = 0 \quad \Omega(\tilde{\pi}) = \mathcal{L}_i dt^i d^{n-1}x + \frac{1}{2} \mathcal{L}_{ij}^\mu dt^i dt^j d^{n-2}x_\mu \dots \quad (2.10)$$

Choosing one particular coordinate t^1 in the definition of the integral (2.7), while keeping all the others fixed, the action may be written as in Eq. (1.1).

When Ω is exact, $\Omega = dQ$, with $Q = Q_0 d^{n-1}x + Q_i^\mu dt^i d^{n-2}x_\mu + \dots$ and Q_0 invariant under G , we recover the invariant actions obtained from invariant Lagrangians, as given by Eq. (2.4). Thus, all invariant actions *that are not associated with invariant Lagrangian*

densities are given by closed forms Ω modulo exact forms dQ , in which the leading terms \mathcal{L}_i and Q_0 respectively are G -invariant.

Two closed differential forms that are continuously connected to each other differ by an exact form ¹⁴. By construction, the form Ω is invariant under reparametrizations of t^i , while under infinitesimal reparametrizations of x it changes by an exact form of the type dQ , which we mod out by. Thus, the class of forms Ω up to exact forms is reduced to that of reparametrization invariant Ω 's, which are now well-defined differential forms on G/H , to be taken modulo forms dQ where Q is also a well-defined form on G/H . Furthermore, since G acts transitively on G/H , the G transform of Ω is continuously connected to the original form as well. Since G is compact, one can construct a G -invariant form by integrating over G with the invariant Haar measure ¹⁴. This has no effect on (2.7) since the leading term \mathcal{L}_i was already G -invariant. Also, one similarly shows that any two continuously connected invariant forms on G/H differ not only by an exterior derivative, but by the exterior derivative of a G -invariant form Q . The space of closed G -invariant well-defined forms on G/H modulo the exterior derivative of G -invariant well-defined forms on G/H can be identified ¹⁴ with the de Rham cohomology of G/H of degree n . Thus, the classification of G -invariant terms in $S[\pi]$ that do not correspond to an invariant Lagrangian density is reduced to finding the n -th de Rham cohomology group $H^n(G/H; \mathbf{R})$ of the homogeneous space G/H . Similar issues were addressed independently from the point of view of equivariant cohomology in Ref. 15.

3. Cohomology of Homogeneous Spaces

There are two ways in which the study of cohomology can be reduced. First, for a product of spaces $K_1 \times K_2$, we have the Künneth formula ¹⁴

$$H^n(K_1 \times K_2; \mathbf{R}) = \sum_{n_1+n_2=n} H^{n_1}(K_1; \mathbf{R}) \wedge H^{n_2}(K_2; \mathbf{R}) \quad (3.1)$$

which gives $H^n(G/H; \mathbf{R})$ in terms of the cohomology of degree 0 up to n of its factors. Second, when a generator of degree n can be written as a linear combination of products of cohomology generators of degree strictly less than n , it is called *decomposable*. Any generator which does not contain any decomposable components is said to be *primitive*. Thus, the entire cohomology can be reconstructed from the primitive generators on the

factor spaces of G/H .

3.1. Cohomology for the case of G/H a Lie group

When G/H is itself a compact Lie group, it factors into simple Lie groups and a number of $U(1)$ factors. Thus the cohomology is obtained from the primitive generators on $U(1)$ and on all simple Lie groups. We denote a primitive generator of degree k by Ω_k with $\Omega_0 = 1$. The list of primitive generators for the classical Lie groups is given by the following catalog⁷, where $H^*(G; \mathbf{R})$ denotes the cohomology ring of G . We have $H^*(U(1); \mathbf{R}) = \wedge\{1, \Omega_1\}$ and

$$\begin{aligned} H^*(SU(N); \mathbf{R}) &= \wedge\{1, \Omega_{2k+1}, k = 1, 2, \dots, N-1\} \\ H^*(SO(2N); \mathbf{R}) &= \wedge\{1, \Omega_{4k-1}, k = 1, 2, \dots, N-1; \Omega'_{2N-1}\} \\ H^*(SO(2N+1); \mathbf{R}) &= \wedge\{1, \Omega_{4k-1}, k = 1, 2, \dots, N\} \\ H^*(Sp(2N); \mathbf{R}) &= \wedge\{1, \Omega_{4k-1}, k = 1, 2, \dots, N\} \end{aligned} \quad (3.2)$$

The primitive generators are obtained from the left-invariant one forms $U^{-1}dU$.

$$\Omega_k = \frac{1}{k!} \text{tr} \{U^{-1}dU\}^k \quad (3.3)$$

Making use of the cyclicity property of the trace and the fact that we have a power of a form of degree 1, we see that $\Omega_k = 0$ whenever k is an even integer. Antisymmetry of $U^{-1}dU$ for $SO(N)$ groups (combined with symplectic conjugation for $Sp(2N)$) furthermore implies that for these groups $\Omega_k = 0$ whenever $k = 1 \pmod{4}$. The doubling of the generator of degree $4M-1$ for $SO(4M)$ is related to the existence of self-duality in those dimensions. Thus, there are *two* non-trivial invariant actions in dimension $d = 4M-2$ for the group $SO(4M)$. The cohomologies of the exceptional groups may also be found in Ref. 7; the degrees of the primitive generators of E_8 for example are 3, 15, 23, 27, 35, 39, 47 and 59 ! In the case of 4 space-time dimensions, $H^5(G; \mathbf{R})$ produces WZW terms : if G is semi-simple with p factors $SU(N_i)$ with $N_i \geq 3$ and all other factors with $H^5 = 0$, we have p different WZW terms, each with an independent coupling constant.

3.2. Differential Calculus on Homogeneous Spaces G/H

It is a fundamental result¹⁴ that the cohomology of homogeneous spaces G/H is given by the classes of closed G -invariant forms on G/H modulo forms that are the exterior

derivative of G -invariant forms on G/H . The space of all G -invariant forms in turn, is easily described in terms of left differentials θ^A , defined by *

$$\theta = \theta^A T^A = U^{-1} dU, \quad [T^A, T^B] = f^{ABC} T^C \quad (3.4)$$

A general form Ω of degree n

$$\Omega = \frac{1}{n!} \omega_{A_1 \dots A_n} \theta^{A_1} \dots \theta^{A_n} \quad (3.5)$$

is G -invariant, provided the coefficients $\omega_{A_1 \dots A_n}$ are constant, vanish whenever one of the indices A_i corresponds to a generator in \mathcal{H} and are invariant under the adjoint action of the group H . This is easily understood in view of the fact that the \mathcal{H} components of θ transform as a gauge field under G , and the tensor ω must vanish on \mathcal{H} to guarantee gauge invariance. We also need differential forms $\Omega_{B_1 \dots B_m}$ of degree n which are tensors of rank m on \mathcal{G} .

In addition, we define four standard operations ¹⁴ acting on these forms, by

$$\begin{aligned} D\Omega_{B_1 \dots B_m} &= d\Omega_{B_1 \dots B_m} + f_{B_1 BC} \theta_B \Omega_{CB_2 \dots B_m} + \dots + f_{B_m BC} \theta_B \Omega_{B_1 \dots B_{m-1} C} \\ i_A \Omega_{B_1 \dots B_m} &= 1/(n-1)! \omega_{B_1 \dots B_m; AA_2 \dots A_n} \theta^{A_2} \dots \theta^{A_n} \\ L_A \Omega_{B_1 \dots B_m} &= f_{AB_1 B} \Omega_{BB_2 \dots B_m} + \dots + f_{AB_m B} \Omega_{B_1 \dots B_{m-1} B} \\ \Delta \Omega_{B_1 \dots B_m} &= L_A (i_A \Omega_{B_1 \dots B_m}) \end{aligned} \quad (3.6)$$

The (covariant) exterior derivative D satisfies $D^2 = 0$ and increases the degree by one unit while leaving the rank unchanged. The interior product i_A satisfies $i_A i_B + i_B i_A = 0$, and increases the rank and lowers the degree by 1. The \mathcal{G} rotation L_A acts as a derivative and its square $L^2 = L_A L_A$ is the quadratic Casimir operator. The operation Δ lowers the degree by 1 while keeping the same rank; it is the adjoint operator of D . The operations D and Δ commute with L_A , while i_A and L_A transform in the adjoint representation of \mathcal{G} . Two additional relations between these operations

$$i_A D + D i_A = -L_A \quad D\Delta + \Delta D = -L^2 \quad (3.7)$$

* Here, T^A are matrices in the representation of U , f are the structure constants of G and θ is a flat connection satisfying the Maurer Cartan equation $d\theta + \theta^2 = 0$.

form the cornerstone of the analysis of the cohomology of G/H .

3.3. Cohomology of General Compact Homogeneous Spaces G/H : an Outline

It is a standard result that the cohomology of G/H can be obtained from the solution of a purely group theoretic problem ⁸. The explicit construction of generators does not seem to be available however, and will be outlined here. A detailed account is given in Ref. 2. As mentioned before, all cohomology generators of degree n , are determined by the primitive generators on the factors of G/H of degrees up to n .

First, the de Rham cohomology of G/H can be identified with the coset of the space of all closed G -invariant forms on G/H by the space of exterior derivatives of G -invariant forms on G/H . The G -invariant forms on G/H are given by Eq. (3.5) where the coefficients $\omega_{A_1 \dots A_n}$ are (1) constant, (2) invariant under the adjoint action of \mathcal{H} , and (3) zero whenever any of the indices A_i corresponds to a generator of \mathcal{H} .

Second, we construct all closed forms obeying (1), including all exact forms of the type (3.5), but ignoring properties (2) and (3) temporarily. To do so, we exhibit a map (the transgression ⁸) that lowers the degree but increases the rank. Let Ω be a closed G -invariant form of rank 0 and degree n ; using the first equation in (3.7), we see that $D(i_A \Omega) = 0$. Applying the second relation in (3.7), we further see that $i_A \Omega$ is exact as a form on G

$$i_A \Omega = D\Omega_A \quad \Omega_A = DM_A - \frac{1}{L^2} \Delta(i_A \Omega) \quad (3.8)$$

The quadratic Casimir operator in Eq. (3.8) is invertible on $i_A \Omega$ since it is invertible on the simple components of \mathcal{G} and since the presence of invariant $U(1)$ components of \mathcal{G} in $i_A \Omega$ does not occur when Ω is a primitive form of degree $n \geq 2$. From Ω_A , we construct a new * form $i_{\{A} \Omega_{B\}}$, which is again closed in view of Eq. (3.7-8). This process may be continued into a hierarchy of equations. For a *primitive generator* of degree n , the hierarchy terminates at $m = [(n-1)/2]$ where forms of degree 0 are encountered.

$$\begin{aligned} D(i_{B_1} \Omega) = 0 &\Rightarrow i_{B_1} \Omega = D\Omega_{B_1}, \\ D(i_{\{B_{k+1}} \Omega_{B_1 \dots B_k\}}) = 0 &\Rightarrow i_{\{B_{k+1}} \Omega_{B_1 \dots B_k\}} = D\Omega_{B_1 \dots B_{k+1}}, \quad 1 \leq k \leq m-1 \\ D(i_{\{B_{m+1}} \Omega_{B_1 \dots B_m\}}) = 0 &\Rightarrow i_{\{B_{m+1}} \Omega_{B_1 \dots B_m\}} = D\Omega_{B_1 \dots B_{m+1}} + d_{B_1 \dots B_{m+1}}. \end{aligned} \quad (3.9)$$

* Curly brackets denote symmetrization of the corresponding indices.

For odd $n = 2m + 1$, $d_{B_1 \dots B_{m+1}}$ is a constant G -invariant tensor, completely symmetric in its indices, and $\Omega_{B_1 \dots B_{m+1}} = 0$. When $H = 1$, we recover the cohomology of Lie groups of §3.1; no extra conditions are needed in this case and the invariant tensor d reproduces the generator of degree $2m+1$ on G , given by Eq. (3.3). When $n = 2m+2$ is even, $d_{B_1 \dots B_{m+1}} = 0$ and $\Omega_{B_1 \dots B_{m+1}}$ is constant and \mathcal{H} -invariant, but it is not necessarily \mathcal{G} -invariant. The analysis of cohomology is thus reduced to the analysis of constant tensors with certain invariance properties, a problem that can be solved in terms of group characters.

Third, property (3) puts restrictions on the allowed symmetric tensors : *the d -symbols must vanish whenever all of its indices correspond to generators of \mathcal{H}* . This follows from a repeated application of the map used in Eq. (3.8)

$$d_{B_1 \dots B_m} = \prod_{i=1}^{m-1} \left\{ -i_{\{B_i} \frac{1}{L^2} \Delta \right\} (i_{B_m} \} \Omega) \quad (3.10)$$

Since the original Ω vanishes on \mathcal{H} , the i_{B_i} to the extreme left will always vanish when all indices B_i correspond to generators in \mathcal{H} . Thus, a non-zero d tensor will correspond to a primitive generator on G/H only if it vanishes on \mathcal{H} .

Finally, one integrates the hierarchy of equations and obtains all closed forms of the type (3.5) obeying properties (1) and (2). Property (3) is imposed explicitly and exact generators are discarded by simple enumeration. The remaining forms are precisely the de Rham cohomology generators of G/H .

4. Special Effective Actions in Dimensions 1, 2, 3 and 4

Cohomology of degree 1 arises when G/H is not simply connected, and the generators are easily found by duality with the homology cycles on G/H . If G/H has product factors of $U(1)$, there will be a generator of degree 1 for each factor.

Primitive cohomology generators of degree 2 are completely determined by the maximal number r of Abelian $U(1)$ factors of H . To each generator T^{α_l} , there corresponds a primitive generator

$$\Omega^{(l)} = -d(\theta^{\alpha_l}) = -W_{\alpha_l} \quad 1 \leq l \leq r \quad (4.1)$$

where W is the field strength associated with the composite \mathcal{H} -valued gauge field V , defined in Eq. (2.3). These generators span the first Chern class of G/H .

Primitive generators of degree 3 are associated with G -invariant symmetric rank 2 tensors d_{BC} on \mathcal{G} which vanish on \mathcal{H} and it is easy to integrate the formalism of §3.3. The primitive generators are Goldstone-Wilczek currents¹⁶, gauged under the subgroup H and given by

$$\Omega = \frac{1}{6}d_{ab}f_{bcd}\theta^a\theta^b\theta^c + \frac{2}{3}d_{a\beta}f_{\beta cd}\theta^a\theta^b\theta^c \quad (4.2)$$

A primitive generator Ω of degree 4 produces a hierarchy in (3.9) where the d tensor is absent, and where $\Omega_{B_1B_2}$ is a constant \mathcal{H} -invariant symmetric tensor on \mathcal{G} . As a form on G , Ω must be exact, since there are no primitive generators of even degree on a Lie group. Applying the procedures of §3.3, we find that

$$\Omega = dQ \quad Q = \frac{1}{2}m_{\beta\lambda}f_{a\beta c}\theta^a\theta^c\theta^\lambda + \frac{1}{6}m_{\beta\lambda}f_{\alpha\beta\gamma}\theta^\alpha\theta^\gamma\theta^\lambda \quad (4.3)$$

or $\Omega = m_{\alpha\beta}W_\alpha W_\beta$. Here, $W_\alpha = d\theta_\alpha + f_{\alpha\gamma\delta}\theta_\gamma\theta_\delta/2$ are the components of the curvature form W and $m_{\alpha\beta}$ is any constant symmetric tensor, invariant under the action of the adjoint representation of \mathcal{H} . These tensors are just the Cartan-Killing forms on the simple components of \mathcal{H} , and arbitrary coefficients on the $U(1)$ components of \mathcal{H} . The full cohomology of degree 4 (in the case where G/H is simply-connected) is given by

$$\Omega = \sum_{k=1}^q m_2^{(k)} W_\alpha^{(k)} W_\alpha^{(k)} + \sum_{l,m=1}^r m_1^{(l,m)} W_{\alpha_l} W_{\alpha_m} \quad (4.4)$$

The generators of $H^4(G/H; \mathbf{R})$ in the first sum belong to the second Chern class evaluated on the \mathcal{H} -valued connection V of (2.3) with components θ^α , while the generators in the second sum arise from products of generators belonging to the first Chern class. The form Q in (4.3) is a linear combination of Chern-Simons invariants in 3 dimensions^{9,10}, evaluated on the composite gauge field V of (2.3). The resulting invariant effective action coincides with the Chern-Simons action evaluated on composite connections.

Cohomology of degree 5 is the one relevant to actions in 4-dimensional space-time. According to the general procedure outlined in §3.3, the primitive generators of degree 5 are associated with a completely symmetric G -invariant tensor d_{ABC} of rank 3 on \mathcal{G} , which must vanish on \mathcal{H} . This tensor is precisely the one encountered in the study of the chiral gauge anomaly¹⁰ for a gauge group G . Conversely, any non-zero G -invariant tensor d_{ABC} , which vanishes on the subalgebra \mathcal{H} produces a unique primitive generator of $H^5(G/H; \mathbf{R})$, given by

$$\Omega = \frac{1}{480} \left\{ d_{a_1bc} f_{ba_2a_3} f_{ca_4a_5} + 7d_{a_1b\gamma} f_{ba_2a_3} f_{\gamma a_4a_5} + 16d_{a_1\beta\gamma} f_{\beta a_2a_3} f_{\gamma a_4a_5} \right\} \theta^{a_1} \theta^{a_2} \theta^{a_3} \theta^{a_4} \theta^{a_5} \quad (4.5)$$

Notice that $d_{\alpha\beta\gamma}$ does not enter, and that for $H = 1$, we recover Ω_5 of (3.3). All other generators of $H^5(G/H; \mathbf{R})$ are decomposable into linear combinations of products of generators of degrees 1, 2, 3 and 4 which were already discussed above. In particular, the coupling of the Goldstone-Wilczek current to an Abelian composite gauge field strength $W_{\alpha i}$ of (4.1) is of this type and the corresponding action may be recast in four dimensions as the Goldstone Wilczek current coupled to an Abelian (composite) gauge field $\theta^{\alpha i}$.

An alternative procedure for obtaining the same primitive generators is already familiar and was used for example in Ref. 1. In terms of the \mathcal{H} -valued gauge field V , the \mathcal{H} -covariant derivative $D^{\mathcal{H}}$ of Eq. (2.3), and the trace representation for the d -symbol, we obtain a form ^{12,13,17} Ω that indeed vanishes on \mathcal{H} , given by

$$\Omega = \sum_j \frac{1}{5!} d_3^j \text{tr}_{\mathcal{G}_j} \left[(U^{-1} D^{\mathcal{H}} U)^5 - 5W (U^{-1} D^{\mathcal{H}} U)^3 + 10W^2 (U^{-1} D^{\mathcal{H}} U) \right] \quad (4.6)$$

Closure of this form is guaranteed by the fact that d_{ABC} vanishes on \mathcal{H} . Also, this generator cannot be decomposed into a sum of products of generators of lower degree that are well-defined on G/H , and thus Ω is *primitive*.

5. Quantization Conditions, fractionalized statistics of solitons

Different interpolations $\tilde{\pi}$ may be topologically inequivalent; there is no natural way of choosing one interpolation above another. The quantum action, however, is allowed to change additively by integer multiples of 2π under change of interpolating map⁶. The dependence of interpolation then becomes invisible in the quantum theory provided the coupling constants entering Ω are suitably quantized.

Precisely which coupling constants must be quantized depends upon the topology of the space-time manifold M_{n-1} and upon the corresponding cohomology generator Ω in $H^n(G/H; \mathbf{R})$. Two interpolations may be viewed⁶ as interpolations on different balls B_n^1 and B_n^2 both with boundary M_{n-1} . If the two interpolations are topologically inequivalent, then the two balls cannot be continuously connected to each other, and the difference $C_n = B_n^2 - B_n^1$ must form a non-contactible cycle C_n inside G/H . As a general rule, quantization of the coupling constant of Ω must occur when the integral of Ω on the difference cycle C_n is non-zero for at least one pair of interpolations. Notice that in our formulation, quantization conditions for WZW terms and Chern Simons terms appear on the same footing.

If space-time is a sphere S^{n-1} , the difference cycle is a sphere $C_n = S^n$, and *only the coupling constants for primitive generators of $H^n(G/H; \mathbf{R})$ must be quantized*. This includes the cases of the WZW terms in two and four dimensions, of the Chern-Simons action for *the semi-simple* factors of \mathcal{H} in three dimensions, and of the first Chern class in 1 dimension which is the original Dirac quantization condition for magnetic monopoles. Coupling constants for decomposable generators do not have to be quantized however. This case includes in four dimensions the coupling of the Goldstone Wilczek current to a composite $U(1)$ field, or the Chern Simons action in three dimensions associated with the Abelian factors in \mathcal{H} . If space-time is not a sphere, then the quantization conditions are different, and coupling constants for decomposable generators may have to be quantized as well.

Further refinements of the quantization conditions may be required depending on additional topological issues. We shall just consider an example in which space-time is a sphere S^4 , with $\pi_4(G) = 0$ and $\pi_4(H) \neq 0$. For all simple groups H we have $\pi_4(H) = 0$, except $\pi_4(Sp(2p)) = \mathbf{Z}_2$. Whenever $\pi_4(H) \neq 0$, H has a *discrete anomaly* [18], and it can be shown that the coupling constant of the corresponding WZW term of $H^5(G/H; \mathbf{R})$ must be quantized in terms of *even* integers to obtain a single-valued path integral ¹.

The presence of the invariant actions discussed above in any dimension of space-time has important physical consequences on the dynamics of extended objects such as solitons. Witten showed in Ref. 6 that the integer coupling constant N of the WZW term in four dimensions can be viewed as the number of colors of an underlying microscopic quark theory. Solitons in the Goldstone field then acquire bosonic or fermionic statistics depending on whether N is even or odd⁶.

Wilczek and Zee showed in Ref. 11 that in three space-time dimensions and $G/H = S^2$, solitons may acquire fractional spin and statistics due to the presence of a Chern-Simons type term. Our formalism allows for a natural extension of their work to the case of any symmetry breaking pattern G/H when H contains a certain number r of commuting $U(1)$ factors. Assuming G simply connected, there will be r different types of solitons, each with its independent conserved charge g_l , $l = 1, \dots, r$, proportional to the space integral of the $U(1)$ field strength W_{α_l} of (4.1). The corresponding effective action (involving only the Chern Simons terms for the Abelian components of H) may be read off directly from

(4.4)

$$S[\pi] = S_0[\pi] + \sum_{l,m=1}^r m_1^{(l,m)} \int_{M_3} \theta^{\alpha_l} d\theta^{\alpha_m} \quad (5.1)$$

Here $S_0[\pi]$ are manifestly G -invariant contributions to the effective action which in particular would guarantee the existence of solitons. In view of the preceding discussion, the coupling constants $m_1^{(l,m)}$ do not have to be quantized if the topology of space-time is that of a 3-sphere. A soliton with charges $g_l = \delta_{l,p}$ acquires a fractional spin proportional to $m_1^{(p,p)}$ through the terms in (5.1). A soliton with charges $g_l = \delta_{l,p} + \delta_{l,q}$ acquires a fractional spin, which depends on $m_1^{(p,p)}$ and $m_1^{(q,q)}$, but also on a mixing term $m_1^{(p,q)} = m_1^{(q,p)}$. A detailed analysis of this fractionalization phenomenon is given in Ref. 3.

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