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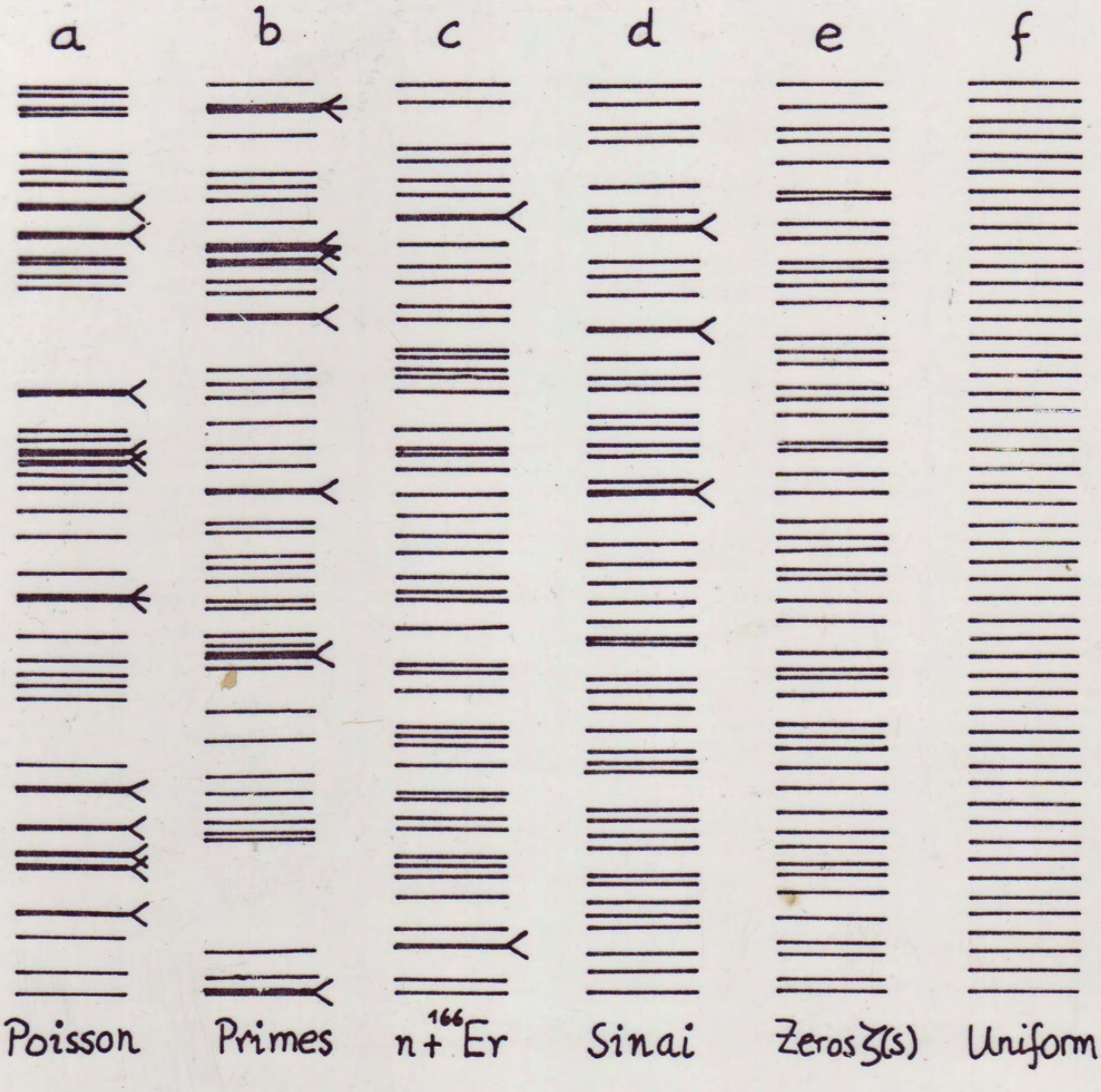
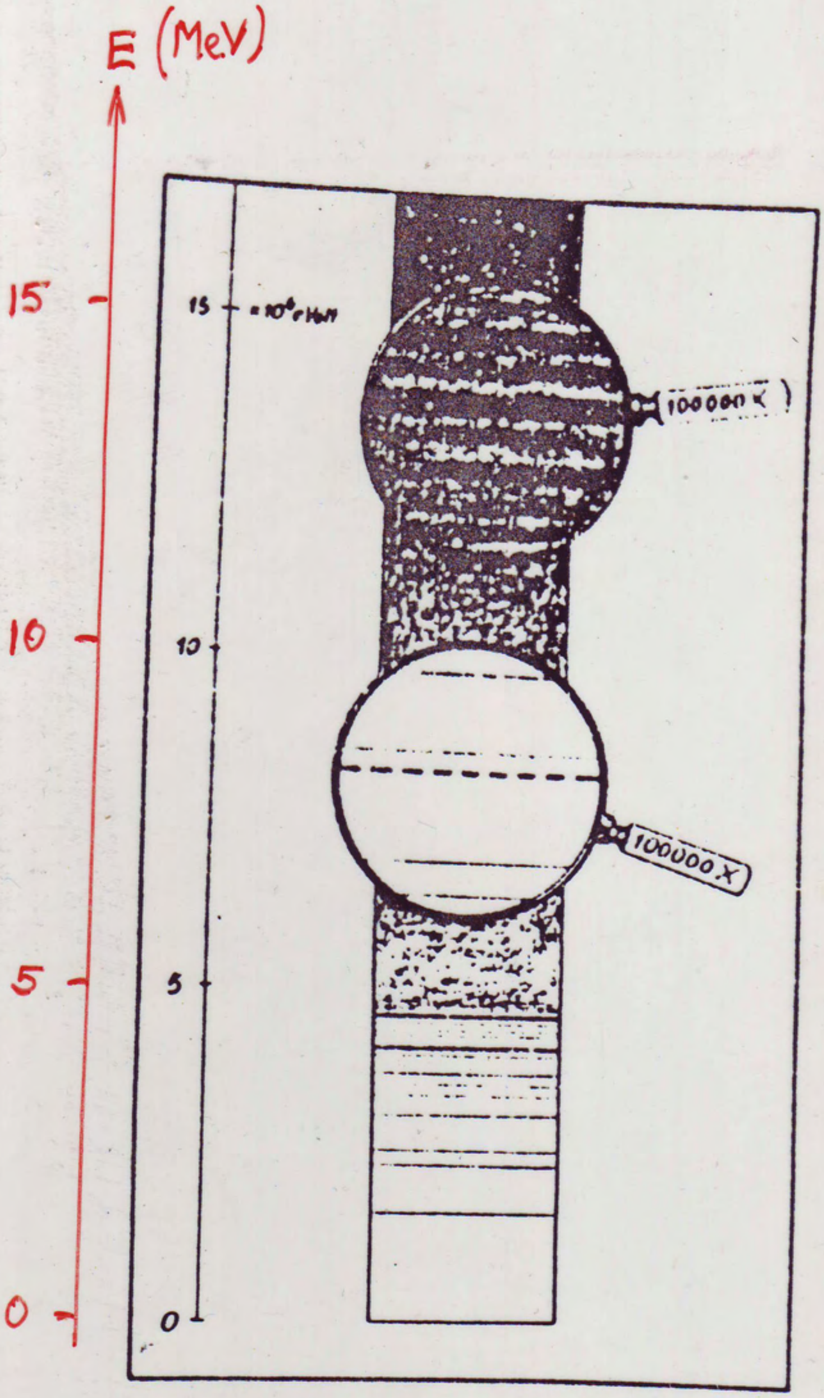


Bohigas Giannoni '84, Lecture Notes in Physics 209



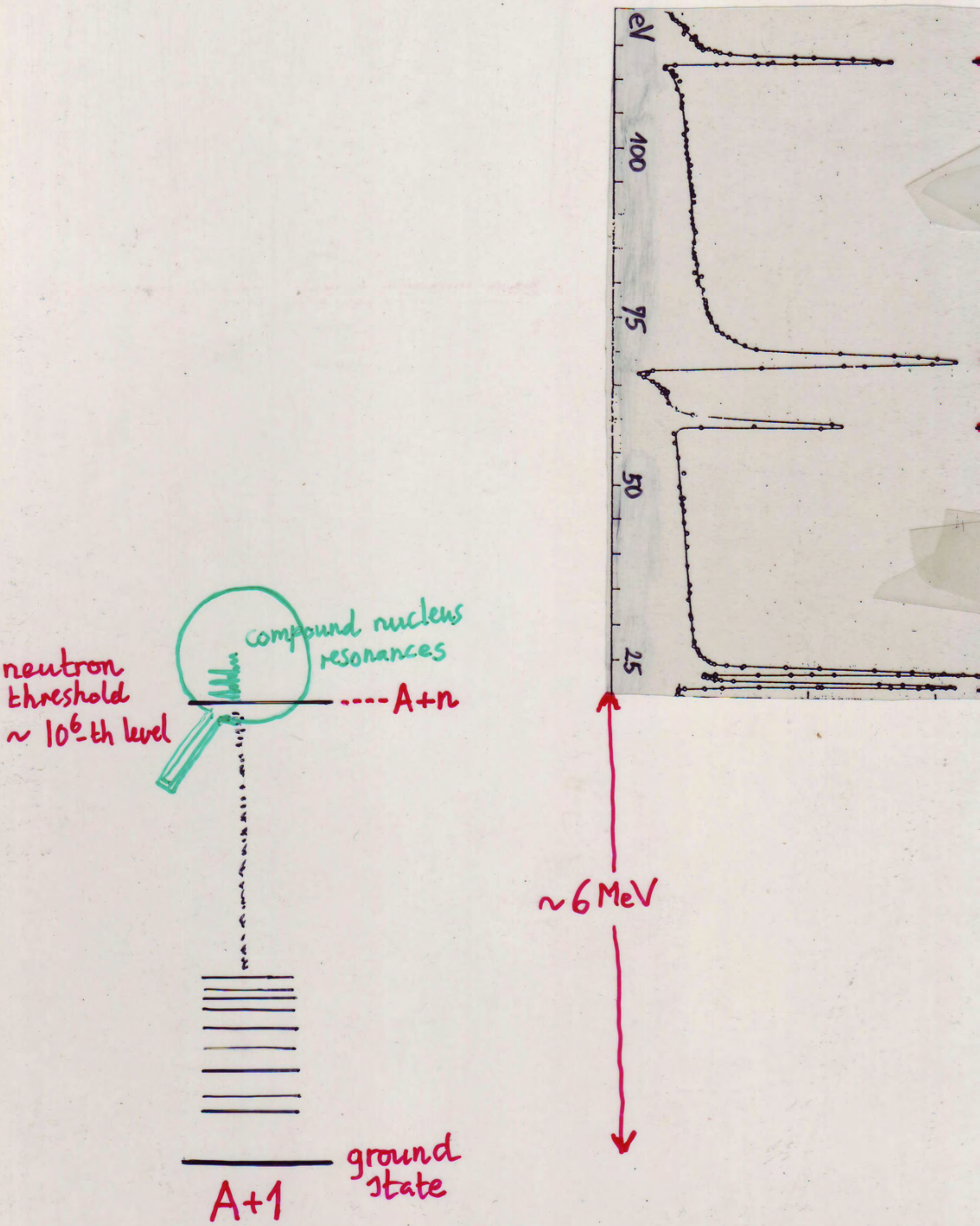
FIGURE

N. Bohr, Nature, Feb. 29, 1936, p. 344
p. 351
'Neutron Capture and Nuclear Constitution'

1.2 Bohr's Concept of the Compound Nucleus

The discovery of narrow resonances in the scattering of slow neutrons by E. Fermi *et al.* [54, 55] and others in the 1930s came as a big surprise to nuclear physicists. It contradicted earlier ideas of independent–particle motion in nuclei and led Bohr to formulate the idea of the "compound nucleus" [22]. Bohr wrote "*In the atom and in the nucleus we have indeed to do with two extreme cases of mechanical many–body problems for which a procedure of approximation resting on a combination of one–body problems, so effective in the former case, loses any validity in the latter where we, from the very beginning, have to do with essential collective aspects of the interplay between the constituent particles*". And: "*The phenomena of neutron capture thus force us to assume that a collision between a ... neutron and a heavy nucleus will in the first place result in the formation of a compound system of remarkable stability. The possible later breaking up of this intermediate system by the ejection of a material particle, or its passing with the emission of radiation to a final stable state, must in fact be considered as separate competing processes which have no immediate connection with the first stage of the encounter*".

Bohr's view was generally adopted. Attempts at developing a theory of


eV
100
75
50
25
compound nucleus
resonances
neutron
threshold
~ 10^6-th level
----A+n
~6 MeV
ground
state
A+1

Neutron cross section σ(E)

$n + {}^{232}Th$

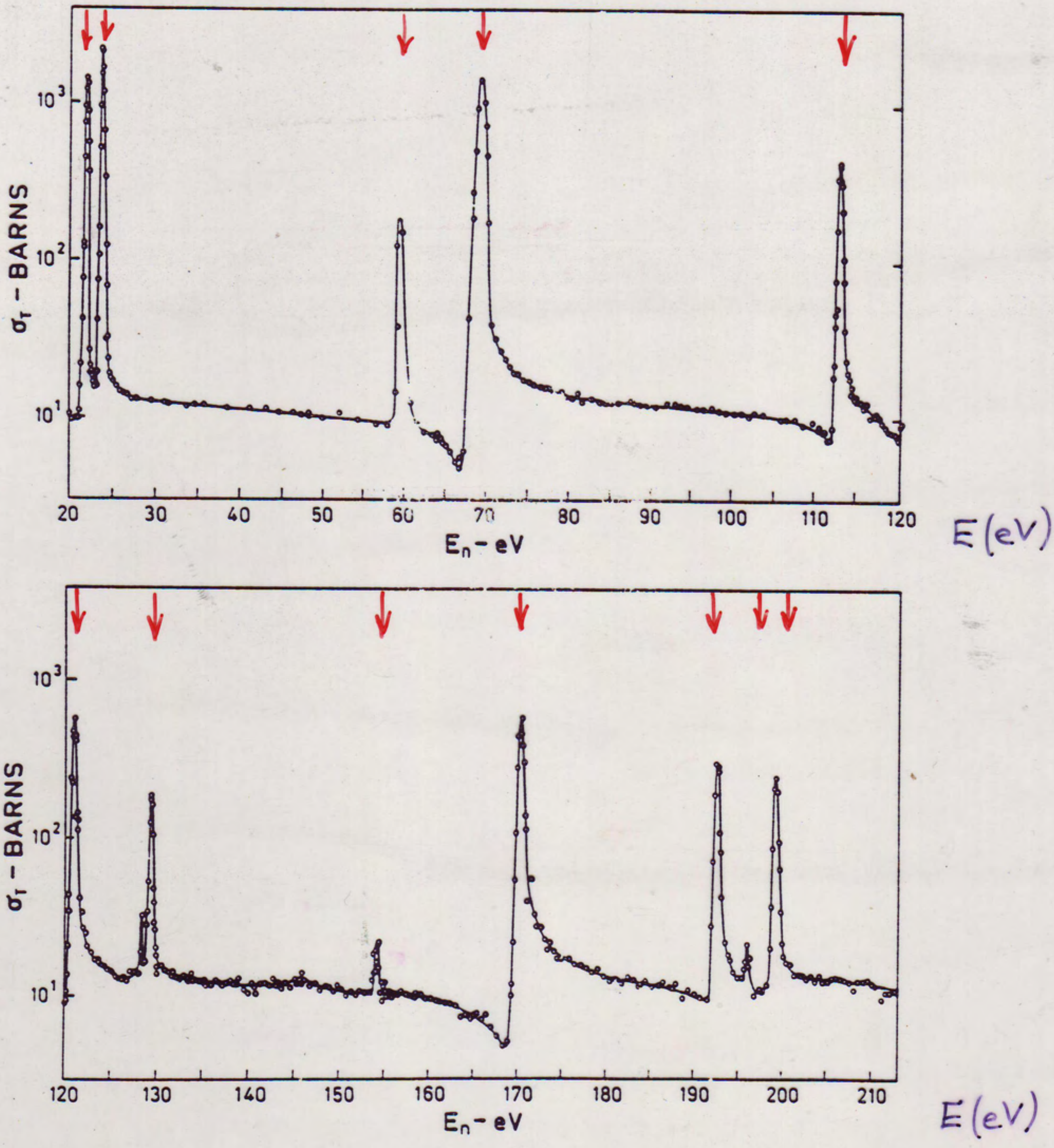
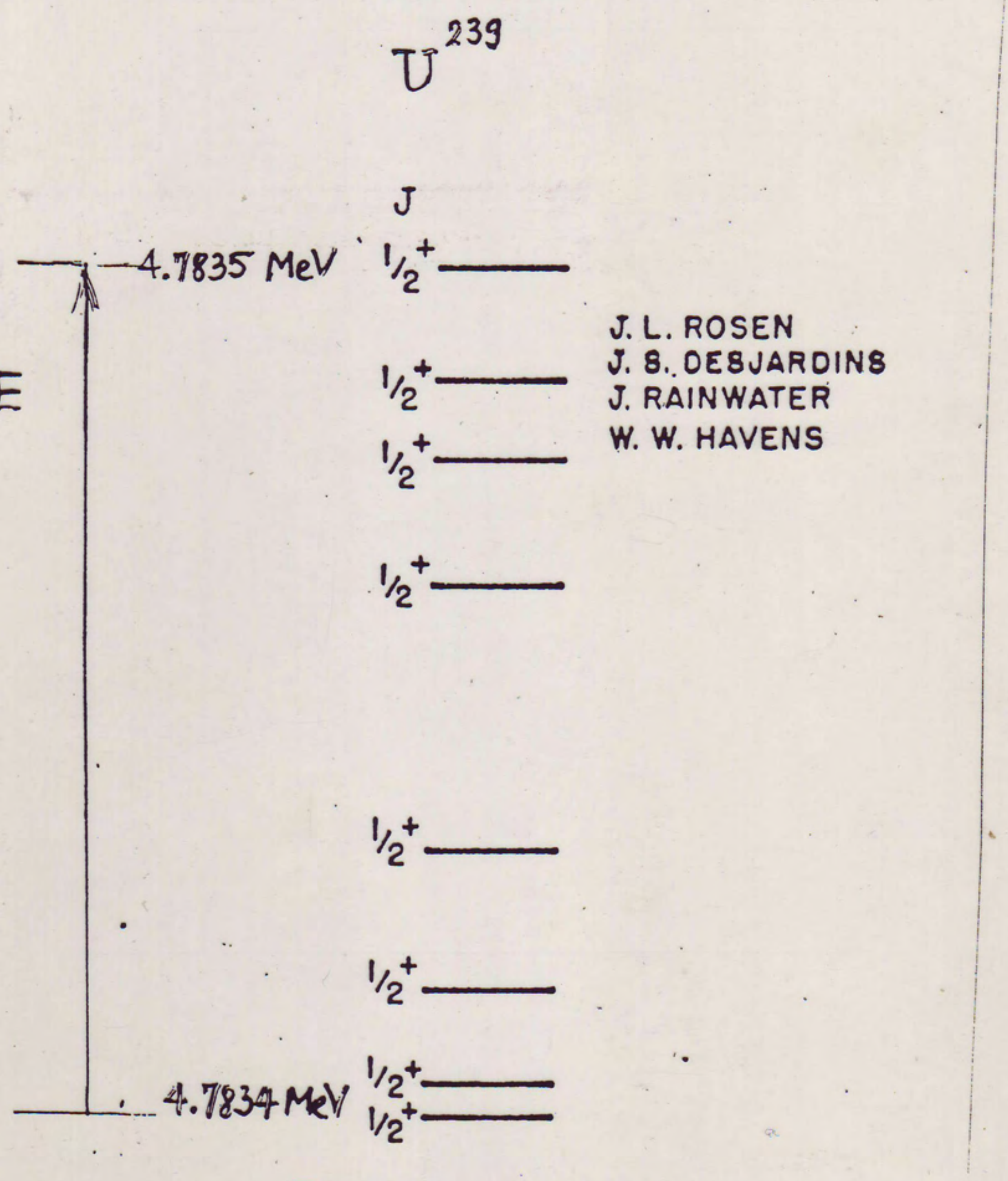


Fig. 3

Compound nucleus resonances



Reproduced from Wigner, *Random Matrices in Physics*

SIAM Review, **9** (1967) 1

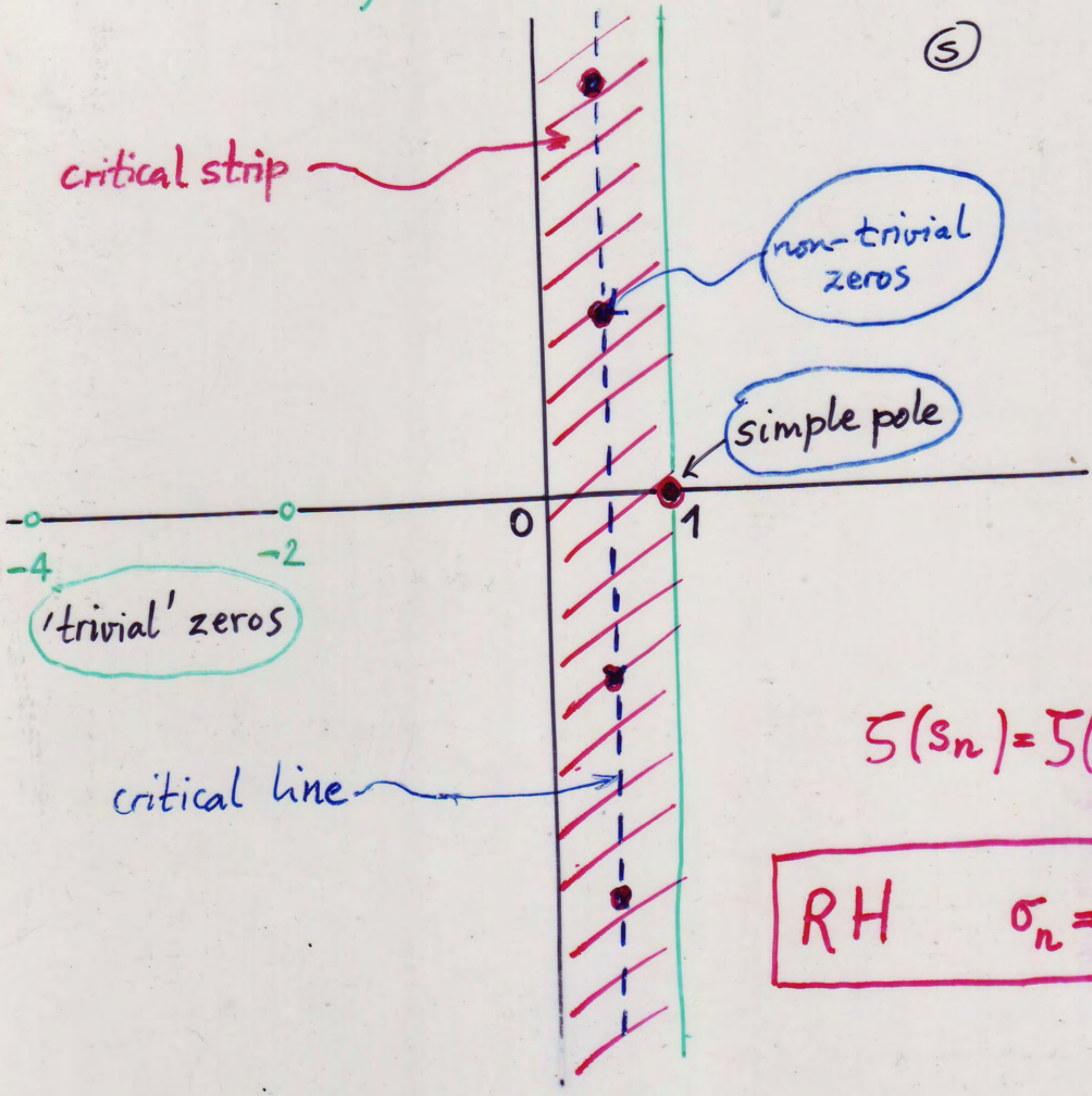
" ... Some physicits myself included, know a few of these energy values by heart because they happen to play an important role in nuclear chain reactors "

Riemann's ζ-function

$$\zeta(s) = \sum_{n} \frac{1}{n^s} = \prod_{p} \left(1 - p^{-s}\right)^{-1}$$

integers (n), primes (p)

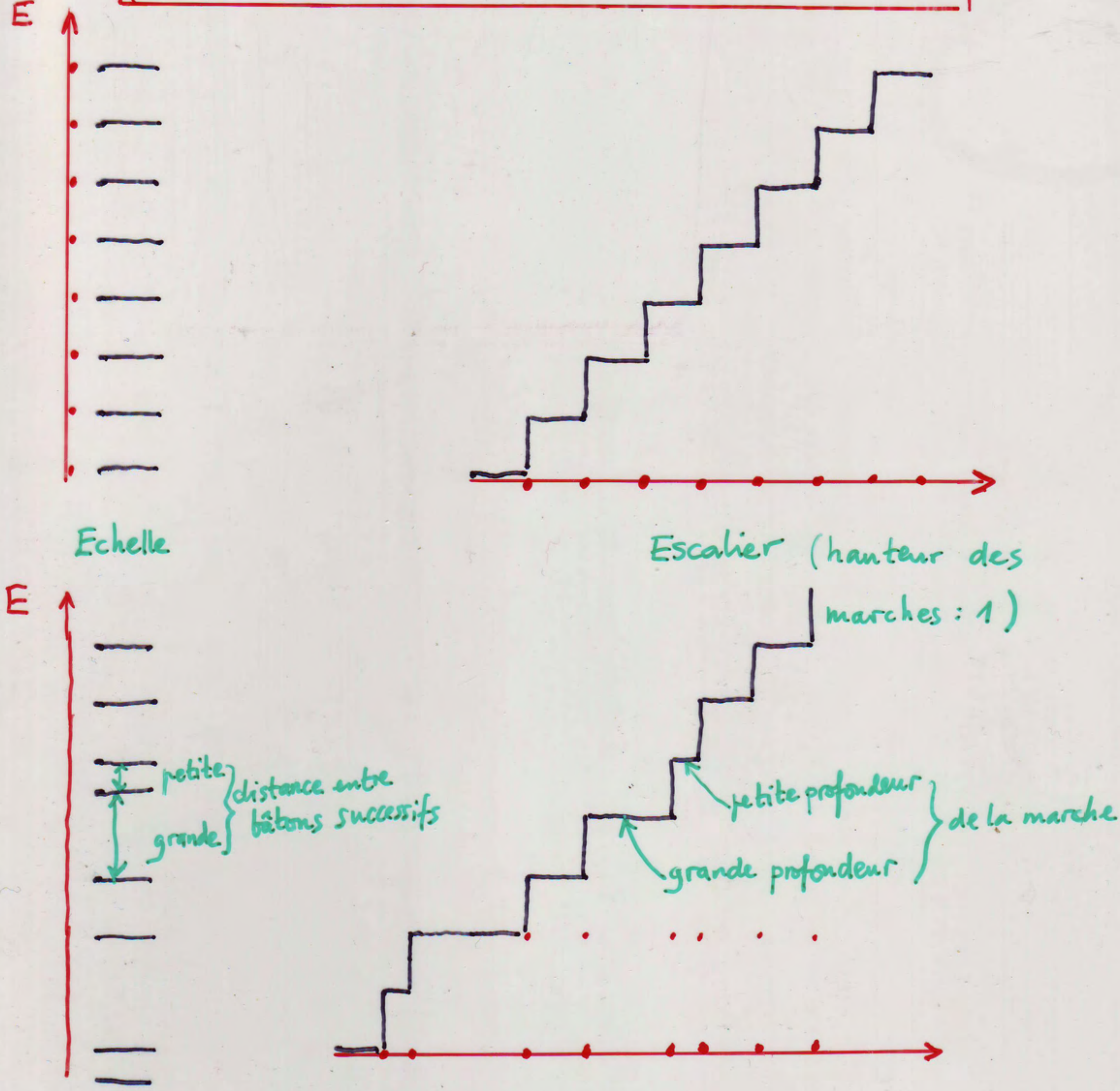
$\mathrm{Re}\, s > 1$



$$\zeta(s_n) = \zeta(\sigma_n + it_n) = 0$$

RH $\sigma_n = \frac{1}{2}$

ECHELLES et ESCALIERS (SPECTRES)



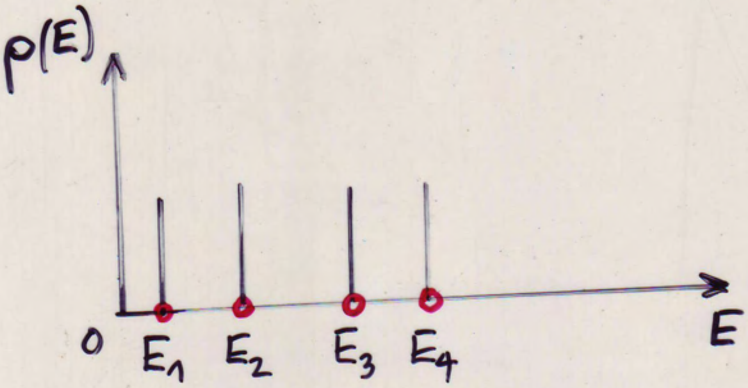
echelle : interdistance entre bâtons successifs

escalier : profondeurs des marches

rugosité (irregularité) des escaliers

density $\rho(E) = \sum_n \delta(E - E_n)$

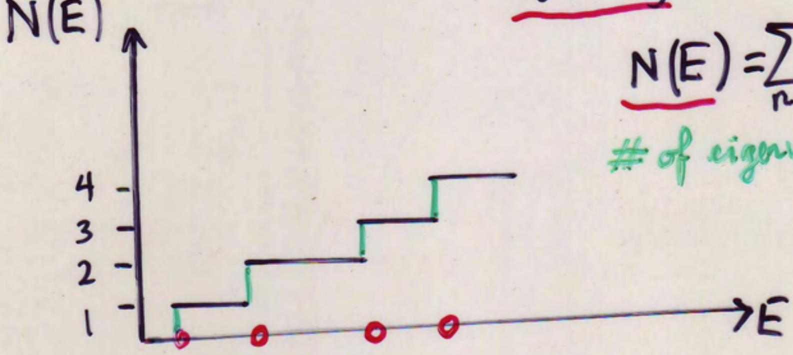
eigenvalues of the system



counting (staircase) function

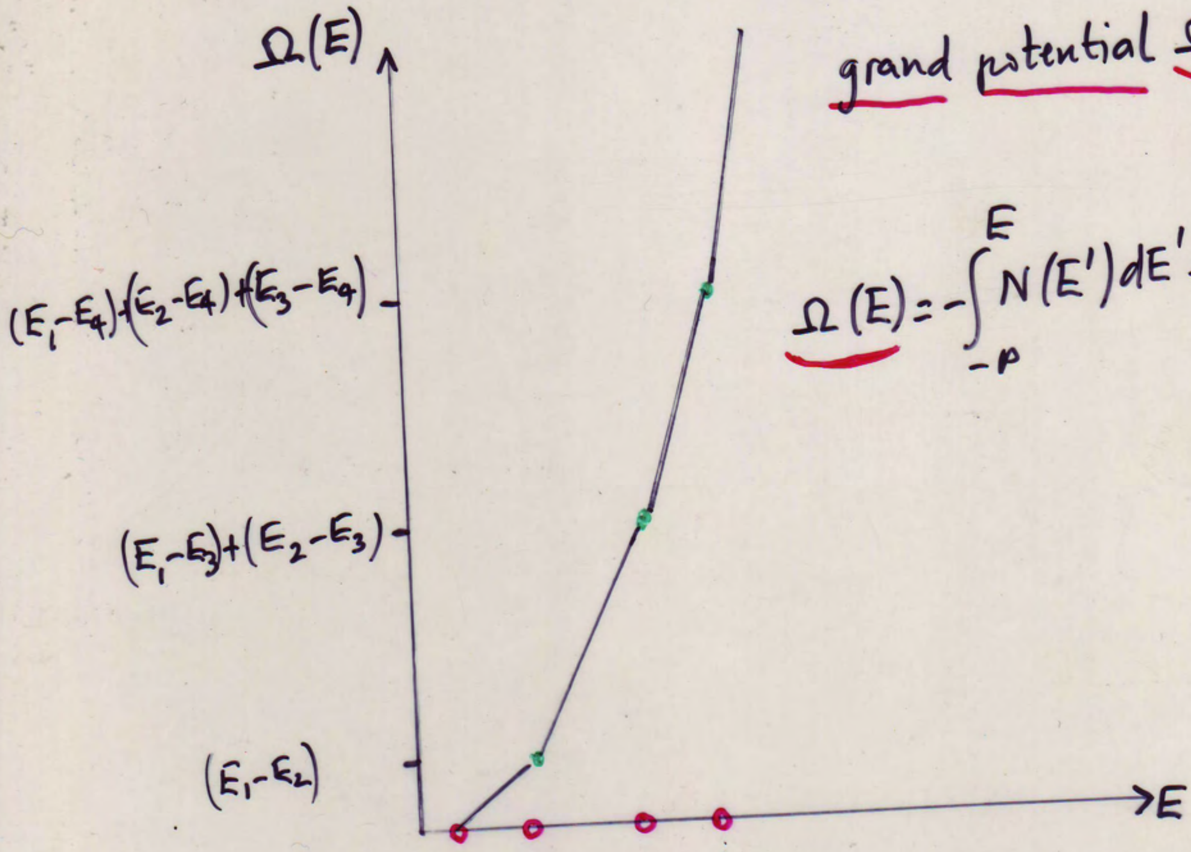
$$N(E) = \sum_n \theta(E - E_n) = \int_{-\infty}^{E} \rho(E')\,dE'$$

# of eigenvalues $\leq E$



grand potential $\Omega(E)$

$$\Omega(E) = -\int_{-\infty}^{E} N(E')\,dE' = \sum_n (E_n - E)\,\theta(E - E_n)$$



General strategy:

To decompose $O(E)$ in a smooth part $\bar{O}(E)$ plus the rest (oscillating part) $\tilde{O}(E)$

$$O(E) = \bar{O}(E) + \tilde{O}(E)$$

O may be the spectral density ρ

counting function N

grand potential Ω

Generally, the smooth part is the easy part

and the oscillating / fluctuating part is the hard part

'The noise is the signal' (R. Landauer)

" ... the Hamiltonian which governs the behavior of a complicated system is a random symmetric matrix, which no particular properties except for its symmetric nature. "

Wigner, in *Symmetries and Reflections*

RANDOM MATRICES

$$\begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1N} \\ H_{21} & H_{22} & \cdots & H_{2N} \\ \vdots & & & \\ H_{N1} & H_{N2} & \cdots & H_{NN} \end{pmatrix}$$

real symmetric $\beta = 1$

hermitian $\beta = 2$

real quaternion $\beta = 4$

H_{ij} : r.i.v $(0, v^2)$ $\qquad \mathcal{P}(H) \propto \exp(-\mathrm{Tr}\, H^2)$

Eigenvalue distribution

$$\mathcal{P}(E_1, E_2, \ldots, E_N) \propto \exp\left\{-\frac{1}{4v^2}\sum E_i^2\right\} \prod_{i<j} |E_i - E_j|^{\beta}$$

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad \text{GOE} \qquad \beta = 1$$

$$\begin{pmatrix} a & x \\ x^* & c \end{pmatrix} \qquad \text{GUE} \qquad \beta = 2$$

$$\begin{pmatrix} a & 0 & x & y \\ 0 & a & -y^* & x^* \\ x^* & -y & c & 0 \\ y^* & x & 0 & c \end{pmatrix} \qquad \text{GSE} \qquad \beta = 4$$

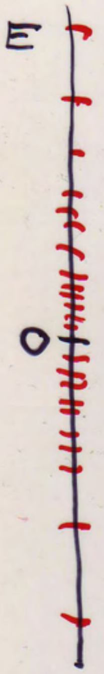
a, b, c real

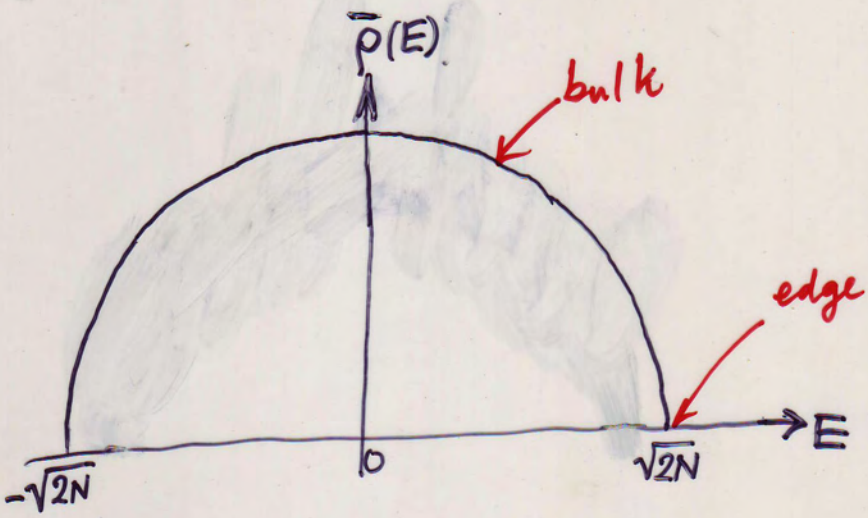
x, y complex

$$P_{N\beta}(H) = K_{N\beta} \exp\left\{ -\mathrm{Tr}(H^2)/4v^2 \right\}$$

v provides the scale

$$\bar{\rho}(E) = \begin{cases} (2\pi\beta v^2)^{-1} \, (4N\beta v^2 - E^2)^{1/2} & |E| < 2\sqrt{\beta v^2 N} \\ 0 & |E| > 2\sqrt{\beta v^2 N} \end{cases}$$





Wigner's semicircle law
for average spectral density

Smooth part of spectral density in RMT

A single parameter in the theory, which fixes, say, the scale

Wigner-Dyson (Gaussian)

H : $N \times N$ hermitean matrix

H : real eigenvalues

$$P(H) \propto e^{-\frac{\beta}{2} \mathrm{Tr}\, H^2}$$

Wishart

X : $M \times N$ matrix $M \geq N$ $\left(\frac{N}{M} = c \leq 1\right)$

$W = X^+ X$: hermitean $N \times N$ positive definite (Wishart) matrix

X^+ : conjugate transpose

$$P(X) \propto e^{-\frac{\beta}{2} \mathrm{Tr}\, X^+ X}$$

X : gaussian entries (i.i.d.)

Joint distribution of eigenvalues

Gaussian

$$\propto e^{-\frac{\beta}{2}\sum_1^N \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta \prod_1^N d\lambda_i$$

Wishart

$$\propto \prod_1^N \lambda_i^{\frac{\beta}{2}(1+M-N)-1} e^{-\frac{\beta}{2}\lambda_i} \cdot \prod_{j<k} |\lambda_j - \lambda_k|^\beta \cdot \prod_1^N d\lambda_i$$

for $c=1$, $\beta=2$

$$\propto \prod_1^N e^{-\lambda_i} \cdot \prod_{j<k} |\lambda_j - \lambda_k|^2 \prod_1^N d\lambda_i$$

Gaussian

2 d Coulomb gas submitted to external quadratic potential

Wishart

2d Coulomb gas confined to positive half line subject to an external linear + logarithmic potential

Eigenvalue density

Gaussian

$$\rho_N(\lambda) = \frac{1}{\sqrt{N}} f\left(\frac{\lambda}{\sqrt{N}}\right)$$

$$f(x) = \sqrt{\frac{1}{\pi}(2-x^2)} \qquad [-\sqrt{2}, \sqrt{2}]$$

↑ Wigner semi-circle

Wishart

$$\rho_N(\lambda) = \frac{1}{N} f\left(\frac{\lambda}{N}\right)$$

Marčenko-Pastur →

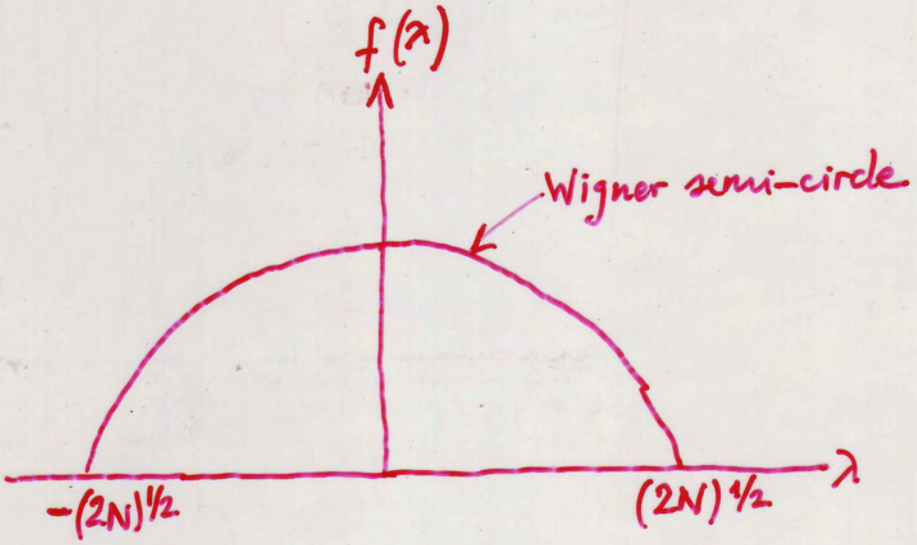
$$f(x) = \frac{1}{2\pi x}\sqrt{(x_+ - x)(x - x_-)}$$

$$x_\pm = \left(\frac{1}{\sqrt{c}} \pm 1\right)^2 \qquad [x_-, x_+]$$

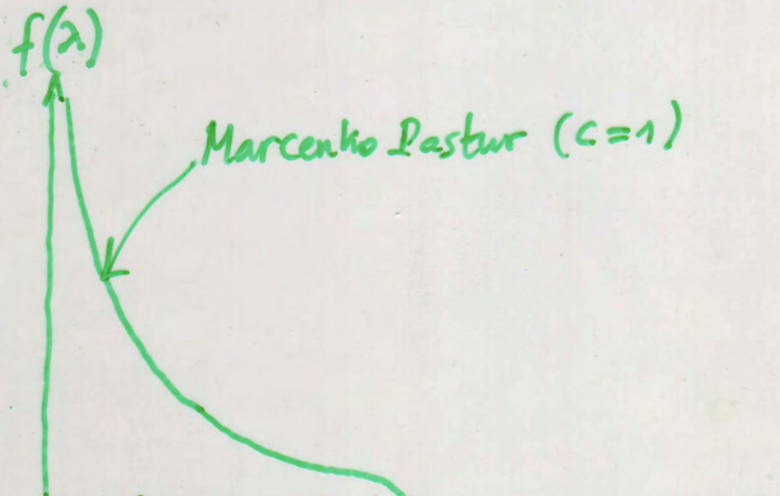
$$c = \frac{N}{M} \leq 1$$

For $c = 1$ $(M = N)$

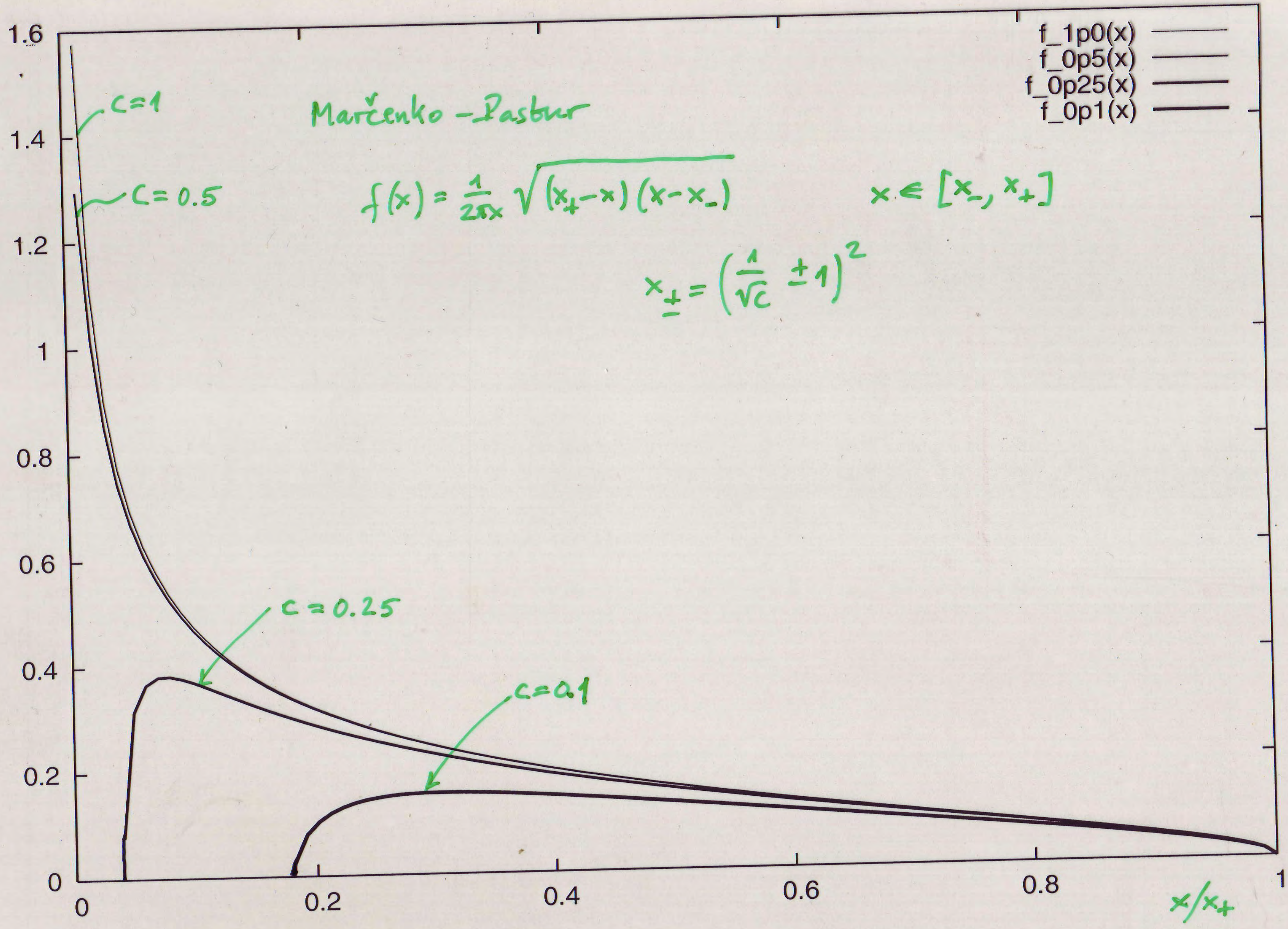
$$f(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}} \qquad [0, 4]$$



f(λ)
Wigner semi-circle
−(2N)^1/2
(2N)^1/2
λ



f(λ)
Marcenko Pastur (c=1)
⟨λ⟩ = N
4N
λ



Marčenko – Pastur
$f(x) = \frac{1}{2\pi x}\sqrt{(x_+ - x)(x - x_-)}$
$x \in [x_-, x_+]$
$x_\pm = \left(\frac{1}{\sqrt{c}} \pm 1\right)^2$
c=1
c=0.5
c=0.25
c=0.1
f_1p0(x)
f_0p5(x)
f_0p25(x)
f_0p1(x)
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
x/x_+

$$\rho(\lambda) = \begin{cases} \frac{1}{\pi}\sqrt{\frac{2}{N}}\sqrt{1-\frac{\lambda^2}{2N}} & |\lambda| \leq (2N)^{1/2} \\ 0 & \end{cases}$$

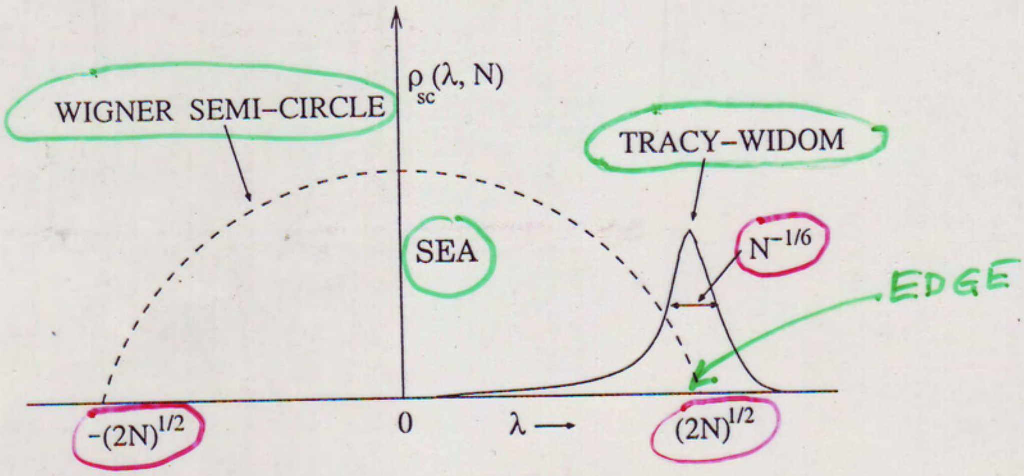


FIG. 1: The dashed line shows the semi-circular form of the average density of states. The largest eigenvalue is centered around its mean $\sqrt{2N}$ and fluctuates over a scale of width $N^{-1/6}$. The probability of fluctuations on this scale is described by the Tracy-Widom distribution (shown schematically).

scaling variable

$$\xi = -\sqrt{2}\, N^{1/6} \left[\lambda_{max} - \sqrt{2N} \right]$$

$\text{Prob}[\xi \leq x] = F_\beta(x)$ ← has a limit $N \to \infty$ (Tracy-Widom)

from Dean, Majumdar PRL 97 (2006) 160201

F: distribution function of largest eigenvalue

Tracy-Widom

β (Dyson index)

1 $$F_1(s)^2 = \exp\left(-\int_s^\infty q(x)\,dx\right)\cdot F_2(s)$$

2 $$F_2(s) = \exp\left(-\int_s^\infty (x-s)\, q(x)^2\,dx\right)$$

4 $$F_4\left(s/\sqrt{2}\right)^2 = \cosh^2\left(\frac{1}{2}\int_s^\infty q(x)\,dx\right)\cdot F_2(s)$$

where q satisfies the Painlevé II equation

$$q'' = xq + 2q^3$$

with boundary condition

Airy function

$$q(x) \sim -Ai(x) \quad \text{as } x \to \infty$$

$$f_\beta(s) = \frac{dF_\beta(s)}{ds}$$

↑ probability density function

$$F_2(s) \to 1 - O\left(\exp\left[-4x^{3/2}/3\right]\right) \qquad x \to \infty$$

$$\to \exp\left[-|x|^3/12\right] \qquad x \to -\infty$$

C. A. Tracy,
H. Widom,
Comm. Math. Phys.
159 (1994) 151
177 (1996) 727

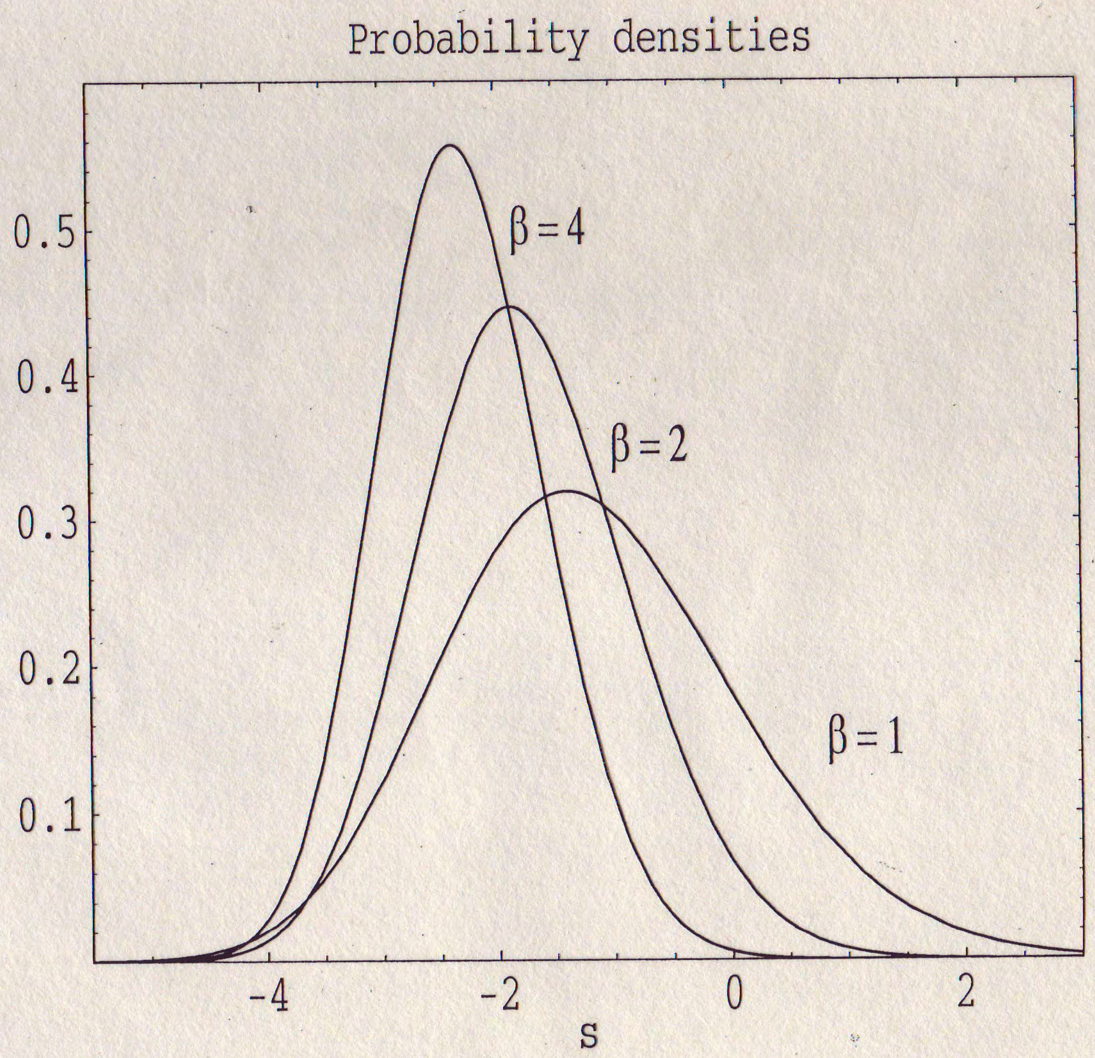


Figure 1: Densities for the scaled largest eigenvalues, $f_\beta(s)$.

LIS Longest increasing subsequence

Sequence of n distinct integers

$X : \{8, 3, 5, 1, 2, 6, 4, 7\}$ $\quad n = 8$

subsequence : an ordered sublist of X : $\{3, 1, 2, 6\}$

longuest increasing subsequence

$\{3, 5, 6, 7\}$, $\{1, 2, 6, 7\}$, $\{1, 2, 4, 7\}$

length of the LIS $\ell = 4$

$X : \{3, 7, 4, 5, 1, 2, 6, 8\}$

LIS $\{3, 4, 5, 6, 8\}$ $\ell = 5$

Consider all $n!$ permutations equally likely (uniform measure)

For each permutation, find ℓ_n

ℓ_n is now a random variable

What is the statistics of ℓ_n? (Ullam's problem, 1961)

Ullam's conjecture $\langle \ell_n \rangle \approx a\sqrt{n}$

$a \approx 2$ ('68)

$a = 2$ (Vershik, 1977)

Longest increasing subsequence in a random permutation of $\{1, \dots, n\}$

L_n

For large n

$$\langle L_n \rangle = 2\sqrt{n}$$

Distribution of

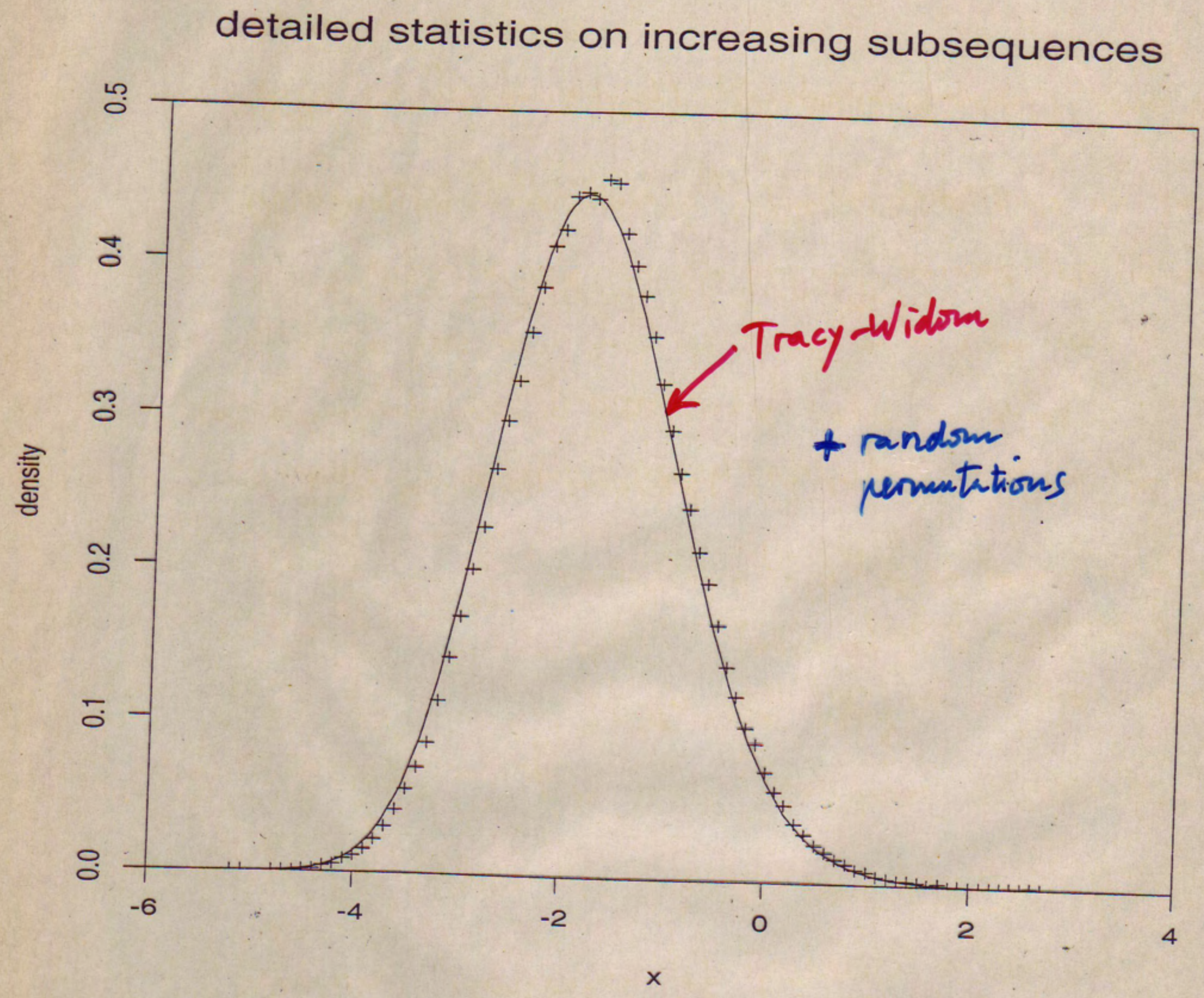
$$(2\sqrt{n} - L_n)/n^{1/6}$$

is Tracy–Widom !! $(\beta = 2)$

RANDOM COMBINATORICS

L_n : length of the longest increasing subsequence of a random permutation of $\{1, \dots, n\}$

$\langle L_n \rangle \sim 2\sqrt{n}$ for n large



$\beta = 2$

Figure 1: Asymptotic density function for for $n = 10^6$. The smooth curve is the asymptotic density function for $(2\sqrt{n} - L_n)/n^{1/6}$, based on theorem of Jinho Baik, Percy Deift, and Kurt Johansson. Data for the asymptotic distribution figure provided by Craig Tracy. Crosses represent the distribution of values of $(2\sqrt{n} - L_n)/n^{1/6}$ for $n = 10^5$ random permutations for $n = 10^6$.

A.M. Odlyzko, E.M. Rains, Preprint 1999

J. Baik, P. Deift, K. Johansson, math.CO/9810105

$$\rho(\lambda) = \frac{1}{N} f\left(\frac{\lambda}{N}\right)$$

$$f(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}} \qquad [0, 4]$$

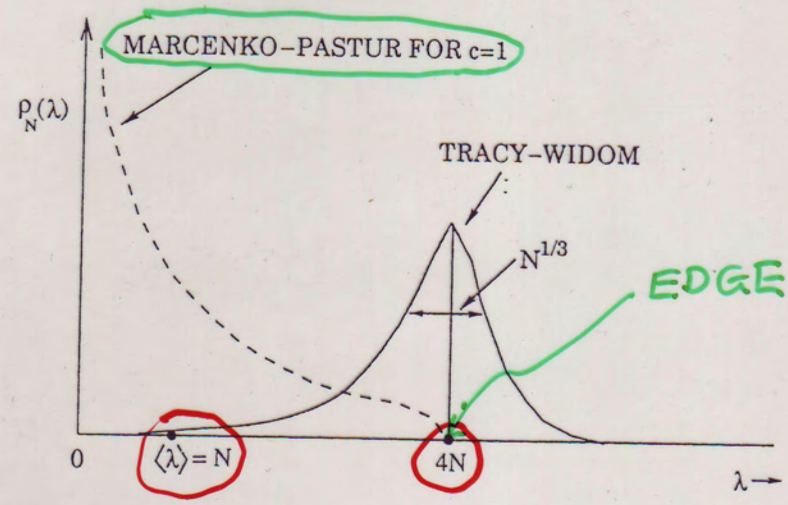


Figure 1. The dashed line schematically shows the Marčenko–Pastur form of the average density of states for $c = 1$. The average eigenvalue for $c = 1$ is $\langle\lambda\rangle = N$. For $c = 1$, the largest eigenvalue is centred around its mean $\langle\lambda_{\max}\rangle = 4N$ and fluctuates over a scale of width $N^{1/3}$. The probability of fluctuations on this scale is described by the Tracy–Widom distribution (shown schematically).

scaling variable

$$\xi = c^{-1/6} x_+^{-2/3} N^{-1/3} (\lambda_{max} - x_+ N)$$

$$x_+ = \left(\frac{1}{\sqrt{c}} + 1\right)^2, \quad c = \frac{N}{M}$$

$\text{Prob}(\xi \leq x) = F_\beta(x)$ has a large N limit (Tracy–Widom)

Johansson
Johnston

From P. Vivo, S. Majumdar, O. Bohigas
cond-mat/0701371
J. Phys. A 40 (2007) 4371

Large deviations and random matrices

Gaussian
Wishart } distribution of λ_{max}

given by Tracy-Widom for typical fluctuations

scale of typical fluctuations

$O(N^{-1/6})$ $\sqrt{2N}$ Gaussian

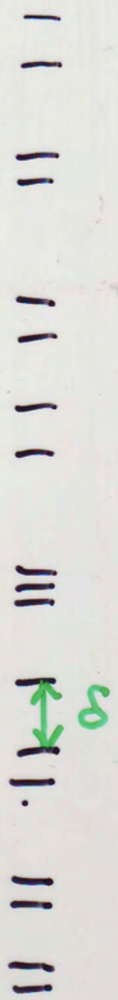
around mean

$O(N^{1/3})$ $x_+(c)N$ Wishart

Study of atypical and large fluctuations of λ_{max} around its mean, over a range

$O(N^{1/2})$ for Gaussian

$O(N)$ for Wishart



Espectre

δ: espaiament entre dos valors propis successius, adjacents

densitat de probabilitat de δ

distribució de primers veïns

$p(s)$

smooth part

$$\begin{cases} \rho = \bar{\rho} + \tilde{\rho} \\ N = \bar{N} + \tilde{N} \\ \Omega = \bar{\Omega} + \tilde{\Omega} \end{cases}$$

~ oscillating part

$$\rho(E) = \sum_i \delta(E - t_i)$$

non-trivial zeros (on critical line)

smooth part

$$\begin{cases} \bar{\rho}(E) = \frac{1}{2\pi} \log \frac{E}{2\pi} + O(E^{-2}) \\ \bar{N}(E) = \frac{E}{2\pi} \log\left(\frac{E}{2\pi}\right) - \frac{E}{2\pi} + \frac{7}{8} + \frac{1}{48\pi E} + O(E^{-3}) \\ \bar{\Omega}(E) = -\frac{E^2}{4\pi} \log\left(\frac{E}{2\pi}\right) + \frac{3}{8\pi} E^2 - \frac{7}{8} E - \frac{\log E}{48\pi} + c + O(E^{-2}) \end{cases}$$

Riemann ζ-function

Proc. London Math. Soc. 3 (1953) 99-117

SOME CALCULATIONS OF THE RIEMANN ZETA-FUNCTION

By A. M. TURING

[Received 29 February 1952.—Read 20 March 1952]

Introduction

In June 1950 the Manchester University Mark 1 Electronic Computer was used to do some calculations concerned with the distribution of the zeros of the Riemann zeta-function. It was intended in fact to determine whether there are any zeros not on the critical line in certain particular intervals. The calculations had been planned some time in advance, but had in fact to be carried out in great haste. If it had not been for the fact that the computer remained in serviceable condition for an unusually long period from 3 p.m. one afternoon to 8 a.m. the following morning it is probable that the calculations would never have been done at all. As it was, the interval $2\pi . 63^2 < t < 2\pi . 64^2$ was investigated during that period, and very little more was accomplished.

The calculations were done in an optimistic hope that a zero would be found off the critical line, and the calculations were directed more towards finding such zeros than proving that none existed. The procedure was such that if it had been accurately followed, and if the machine made no errors in the period, then one could be sure that there were no zeros off the critical line in the interval in question. In practice only a few of the results were checked by repeating the calculation, so that the machine might well have made an error.

FLUCTUATION MEASURES

- Nearest neighbour spacing distribution

$$p(s)$$

- Density-density correlation function
  two-point correlation function
  two-level cluster function

$$Y_2(x)$$

- Its Fourier transform (two-level form factor)

$$b(k) = \int_{-\infty}^{\infty} Y_2(x)\, e^{i2\pi x}\, dx$$

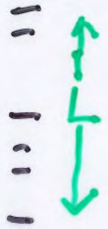
- $$\hat{R}_k(L) = \int_0^L \cdots \int_0^L R_k(x_1, \ldots, x_k)\, dx_1 \ldots dx_k$$

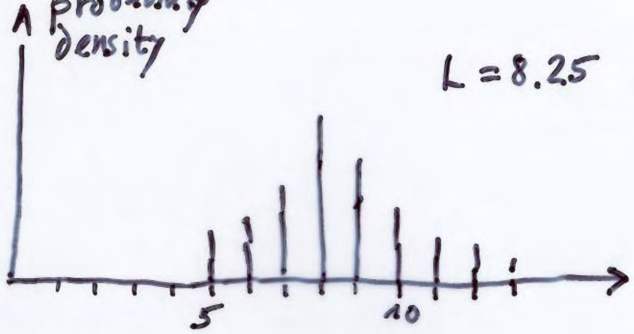
  For small r

  $\dfrac{\hat{R}_k(L)}{k!}$: probability that interval of length L contains k levels

number statistic $n(L)$

$n(L)$: # of points in interval of length L taken at random





- average $\langle n(L) \rangle = L$

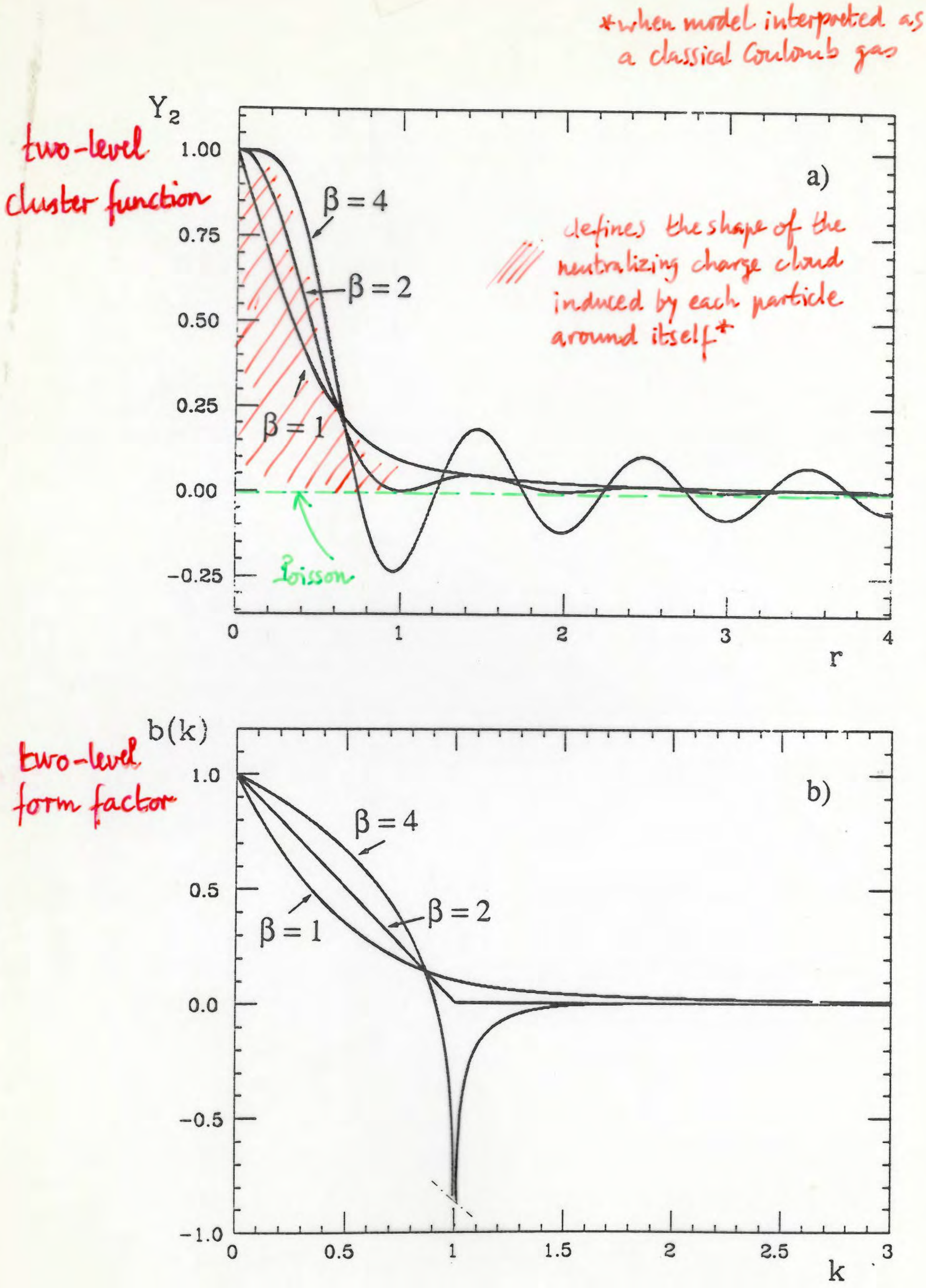
- variance (number variance)

$$\Sigma^2(L) = \langle (n(L) - L)^2 \rangle$$
$$= L - 2\int_0^L (L-r)\, Y_2(r)\, dr = 2\int_0^\infty \frac{\sin^2 \pi L k}{(\pi k)^2} (1 - b(k))\, dk$$

- skewness

$$\gamma_1(L) \cdot \Sigma^3(L) = \langle (n(L) - L)^3 \rangle$$

- excess $$\gamma_2(L) \cdot \Sigma^4(L) = \langle (n(L) - L)^4 \rangle - 3\Sigma^4(L)$$



*when model interpreted as a classical Coulomb gas
Y_2
two-level cluster function
1.00
0.75
0.50
0.25
0.00
−0.25
a)
β = 4
β = 2
β = 1
defines the shape of the neutralizing charge cloud induced by each particle around itself*
Poisson
0
1
2
3
4
r
b(k)
two-level form factor
1.0
0.5
0.0
−0.5
−1.0
b)
β = 4
β = 2
β = 1
0
0.5
1
1.5
2
2.5
3
k

Fig. 6

$$P_N(E_1, \ldots, E_N) = A_N \exp\left\{-U/kT\right\}$$

$$U = \frac{1}{2}\sum E_i^2 - \sum_{i<j} \ln|E_i - E_j|$$

$$P_N(E_1, \ldots, E_N) = C_{N\beta} \exp\left\{-\frac{1}{4v^2}\sum E_i^2\right\} \times$$

$$\times \prod_{i<j} |E_i - E_j|^\beta$$

Let *

$$x_1, x_2, \ldots, x_N$$

be the positions of N points on the real axis, with average density unity and let

$$P_N(x_1, x_2, \ldots, x_N)\, dx_1 \ldots dx_N$$

be the probability, regardless of labelling, of having one point at x_1, another at x_2, ..., another at x_N within each of the intervals $[x_i, x_i + dx_i]$

The statistical properties of the sequence $\{x_i\}$ are characterized by the set of the n-level correlation functions $R_n(x_1, \ldots, x_n)$

$$R_n(x_1, \ldots, x_n) = \frac{N!}{(N-n)!} \int dx_{n+1} \ldots dx_N \, P_N(x_1, \ldots, x_N)$$

From the definition

$$\int R_{n+1}(x_1, \ldots, x_{n+1})\, dx_{n+1} = (N-n)\; R_n(x_1, \ldots x_n)$$

$$Y_k(x_1,\ldots,x_k) = \sum_G (-)^{k-m}\,(m-1)!\,\prod_{j=1}^{m} R_{G_j}(x_t,\ \text{with } t \text{ in } G_j)$$

G stands for any division of the indices $[1,2,.,k]$ into subgroups $[G_1, G_2, \ldots, G_m]$.

k-level cluster functions Y_k :

obtained from R_k by subbstracting out the lower-order correlation terms

Example $$Y_2(x_1,x_2) = -R_2(x_1,x_2) + R_1(x_1)R_2(x_2)$$

$$Y_3(x_1,x_2,x_3) = R_3(x_1,x_2,x_3)$$
$$- [R_1(x_1)R_2(x_2,x_3) + R_1(x_2)R_2(x_1,x_3) + R_1(x_3)R_2(x_1,x_2)]$$
$$+ 2R_1(x_1)R_1(x_2)R_1(x_3)$$

In Statistical Mechanics

Probability distribution for observing the system at the configuration space point $\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_N$ is

$$P_N(\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_N) = \frac{\exp\{-\beta\, U(\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_N)\}}{\int d\underset{\sim}{r}_1 \ldots d\underset{\sim}{r}_N \exp\{-\beta\, U(\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_N)\}}$$

$\beta = 1/kT$

U, for instance

$$U(\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_N) = \sum_{1 \le i < j \le N} v(r_{ij}) \qquad r_{ij} = |\underset{\sim}{r}_i - \underset{\sim}{r}_j|$$

k-point correlation functions $\rho_k(\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_k)$

$$\rho_k(\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_k) = \frac{N!}{(N-k)!} \int d\underset{\sim}{r}_{k+1} \ldots d\underset{\sim}{r}_N\, P_N(\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_N)$$

They satisfy the set of coupled partial differential eqs.

$$\underset{\sim}{\nabla}_1 \rho_k(\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_k) = -\beta \rho_k(\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_k) \sum_{i=2}^{k} \underset{\sim}{\nabla}_1 v(r_{1i}) -$$

$$-\beta \int \rho_{k+1}(\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_{k+1})\, \underset{\sim}{\nabla}_1 v(r_{1\,k+1})\, d\underset{\sim}{r}_{k+1}$$

$$g(\mathbf{r}_1,\mathbf{r}_2) = \rho_2(\mathbf{r}_1,\mathbf{r}_2)/[\rho_1(\mathbf{r}_1)\rho_1(\mathbf{r}_2)]$$

For isotropic fluid: $g(r_{12})$, $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$

↑ radial distribution function

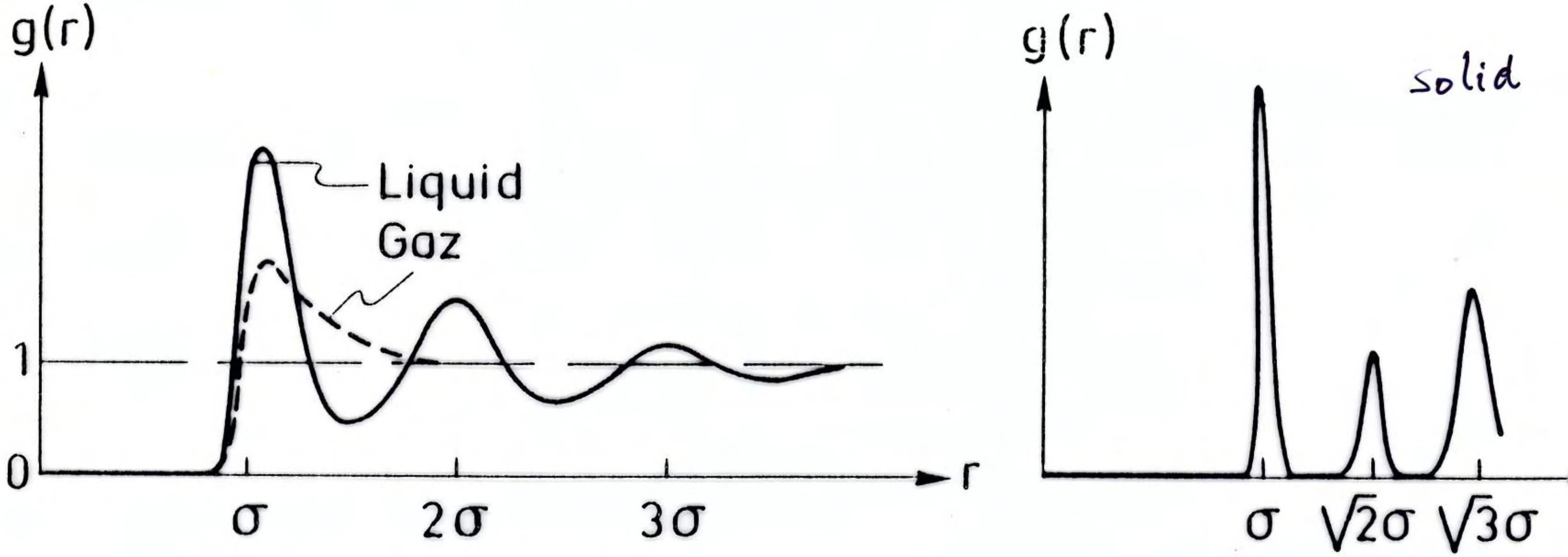
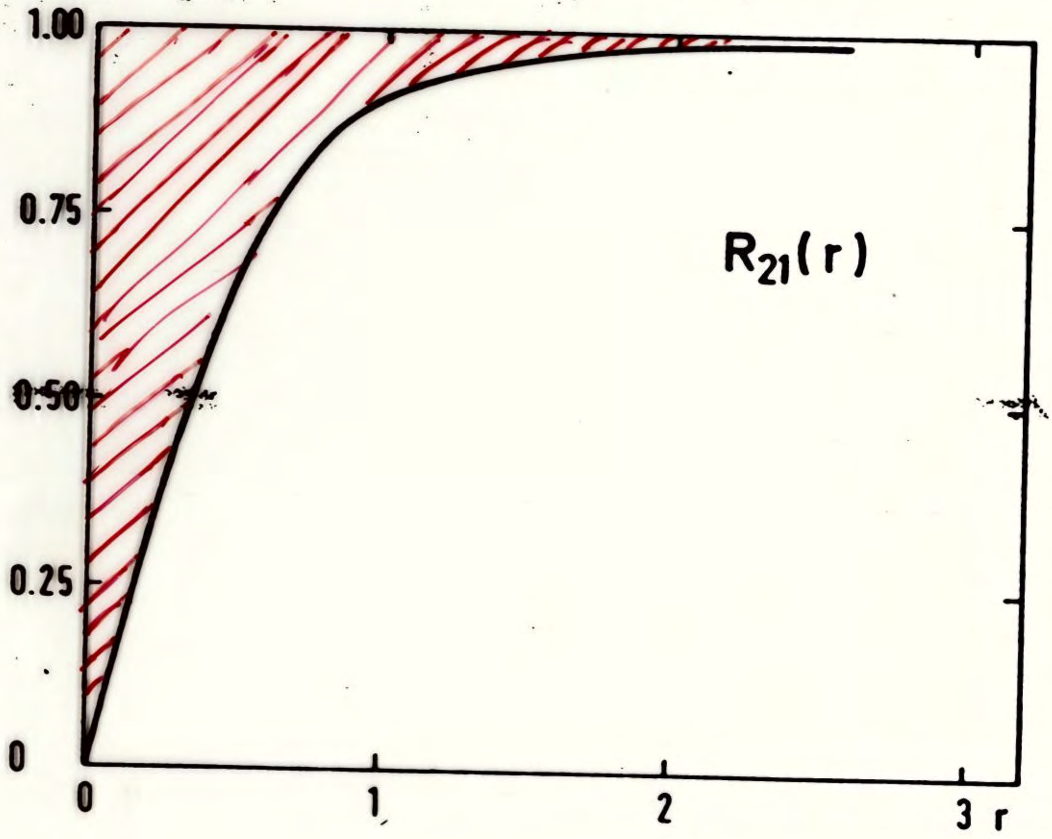
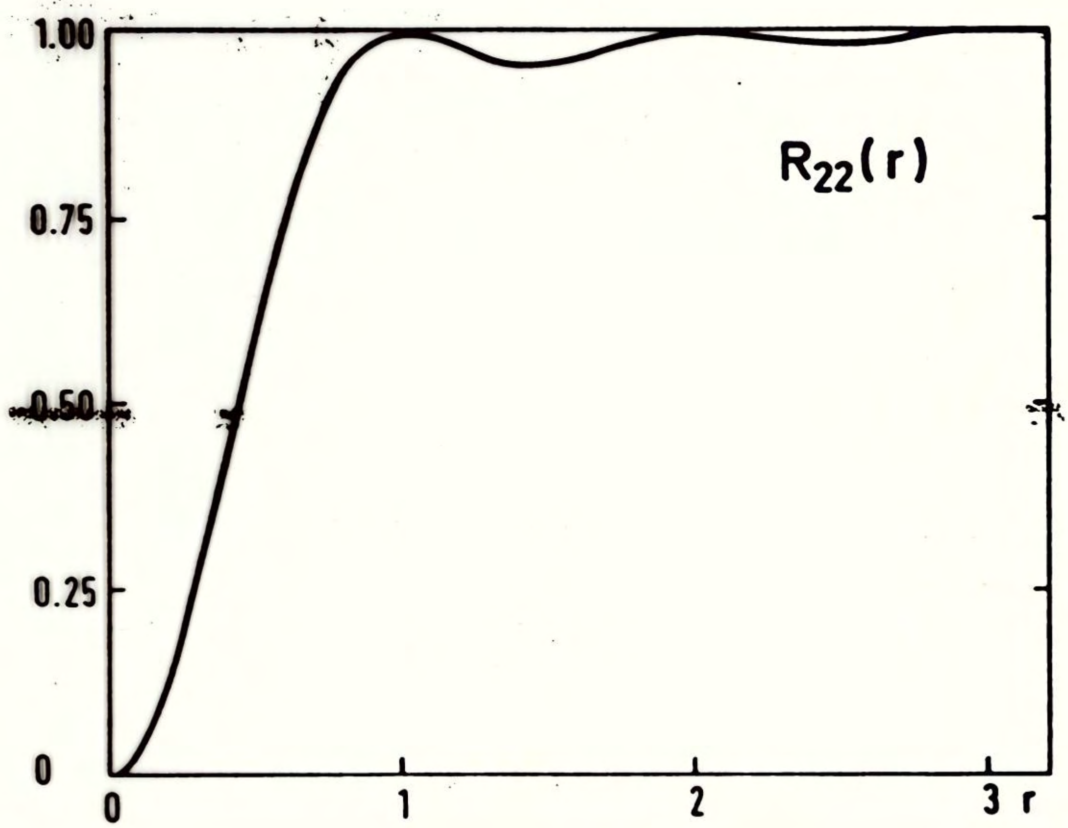


Fig. 2

Y_2 : Defines the shape of the neutralizing charge cloud induced by each particle around itself when the model is interpreted as a classical Coulomb gas

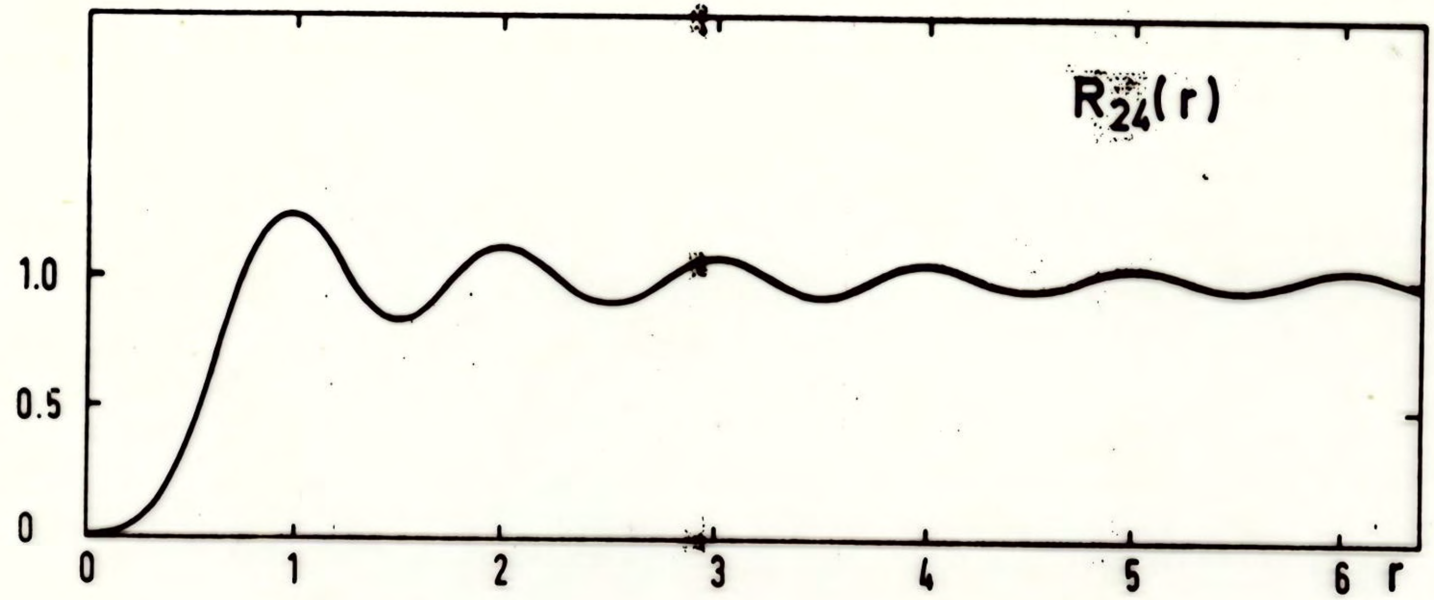


OE $\beta = 1$



$\beta = 2$ UE

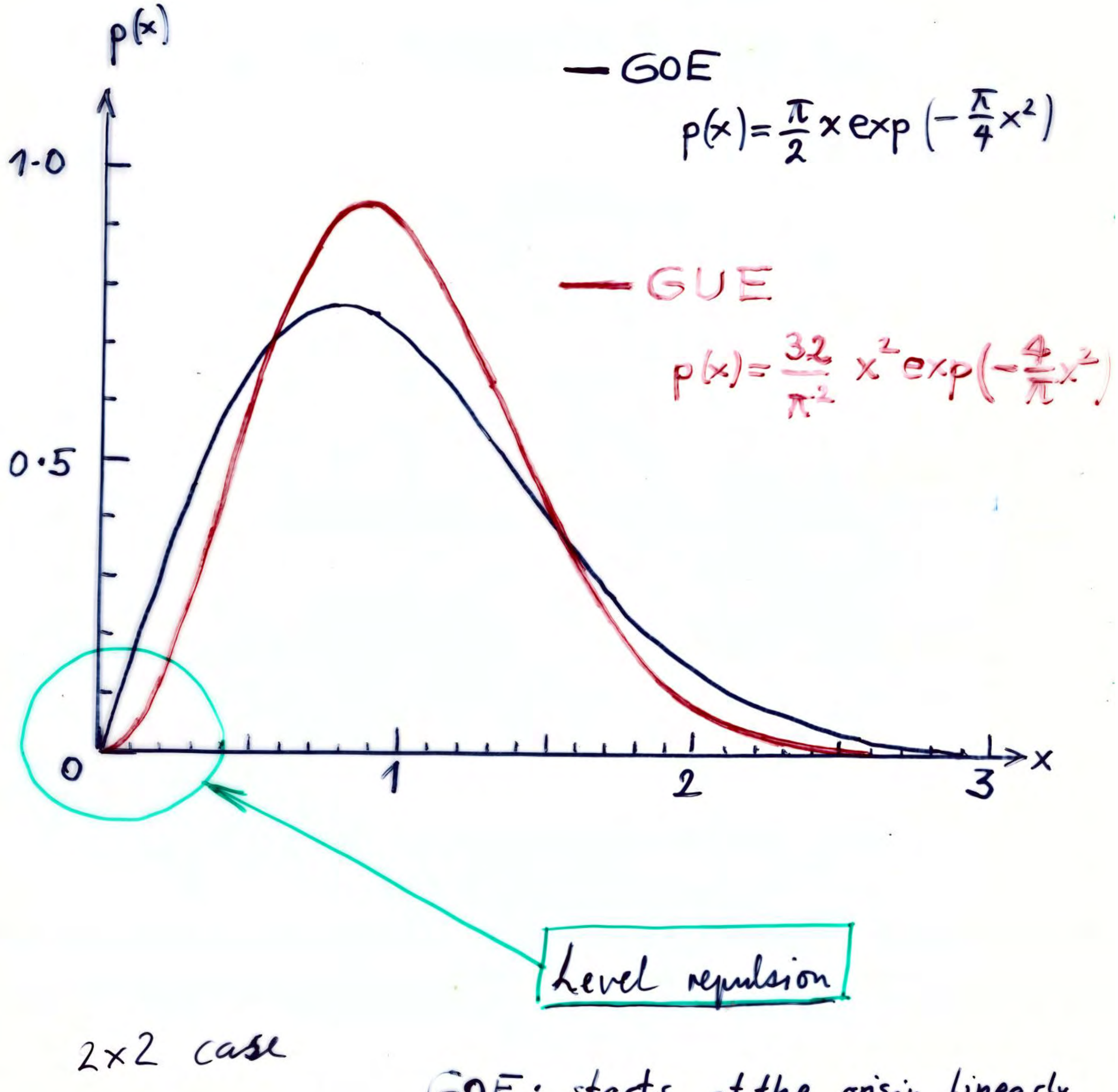
$Y_2(r) = 1 - R_2(r)$



β=4 SE

1.1

Spacing distribution



2×2 case

GOE: starts at the origin linearly

GUE: starts at the origin quadratically

	GOE $p(x=0)$	GOE $p'(x=0)$	GUE $p(x=0)$	GUE $p'(x=0)$	GUE $p''(x=0$
$N=2$	0	$\frac{\pi}{2}$	0	0	$\frac{64}{\pi^2}$
$N\to\infty$	0	$\frac{\pi^2}{6}$	0	0	$\frac{2\pi^2}{3}$

$$\frac{p'(N=2)}{p'(N\to\infty)} = \frac{3}{\pi} = 0.955$$

$$\frac{p''(N=2)}{p''(N\to\infty} = \frac{192}{2\pi^4} = 0.986$$

Small-x of Y_2

$\beta=1$	$\beta=2$	$\beta=4$
$1-\frac{\pi^2}{6}x$	$1-\frac{\pi^2}{3}x^2$	$1-\frac{(2\pi x)^4}{135}$

Small-k of $b(k)$

$\beta=1$	$\beta=2$	$\beta=4$
$1-2k$	$1-k$	$1-\frac{1}{2}k$

Large-L of $\Sigma^2(L)$ (or $\bar{\Delta}(L)$ - factor 2)

$\beta=1$	$\beta=2$	$\beta=4$
$\frac{2}{\pi^2}\ln 2\pi L$	$\frac{1}{\pi^2}\ln 2\pi L$	$\frac{1}{2\pi^2}\ln 4\pi L$

Long range order

$$\int_{-\infty}^{+\infty} Y_2(s)\,ds = 1$$

the spectrum is incompressible

for $\beta=1, 2$ and 4

BEYOND 2-POINT CORRELATIONS

Small-r behaviour of $\hat{R}_2$, $\hat{R}_3$, $\hat{R}_4$

	GOE	UW	Poisson
$\hat{R}_2(r)$	$\frac{\pi^2}{18} r^{3}$	$\frac{\pi}{6} r^{3}$	r^2
$\hat{R}_3(r)$	$\frac{\pi^4}{1350} r^{6}$	! $\frac{\pi^2}{80} r^{5}$	r^3
$\hat{R}_4(r)$	$\frac{\pi^8}{6615000} r^{10}$	!! $\frac{\pi^3}{1680} r^{7}$	r^4

$$\hat{R}_k(r) = \int_0^r \cdots \int_0^r R_k(r_1, \dots r_k)\, dr_1 \dots dr_k$$

For small r, $\hat{R}_k(r)/k!$ gives the probability that there are k levels in an interval of length r

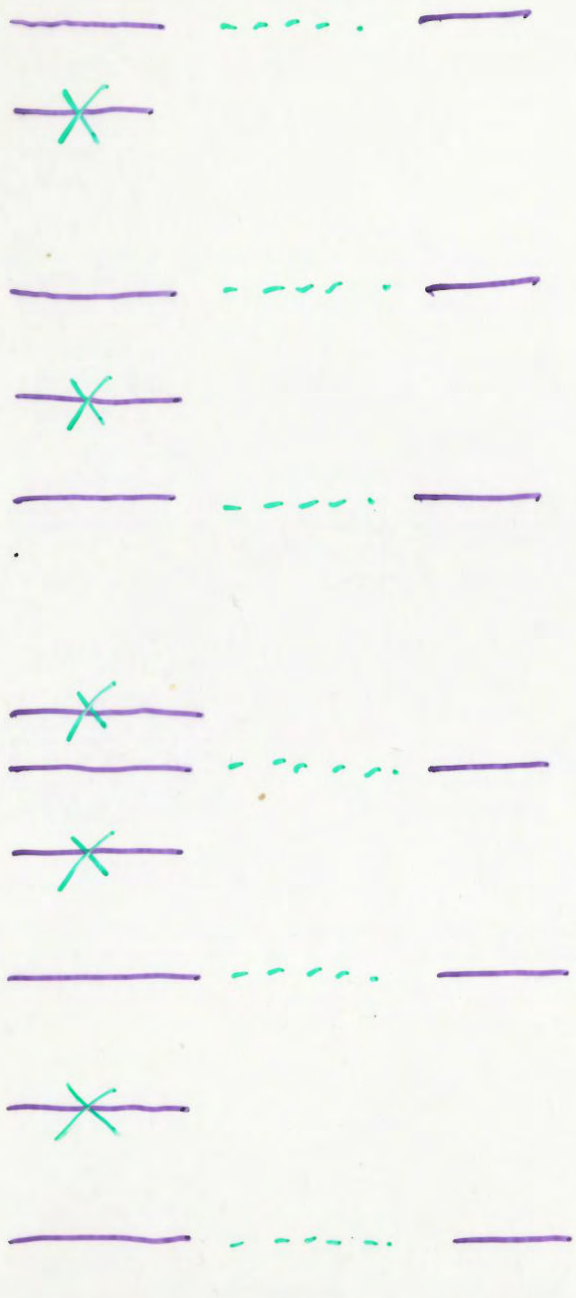
Bohigas, Haq, Pandey

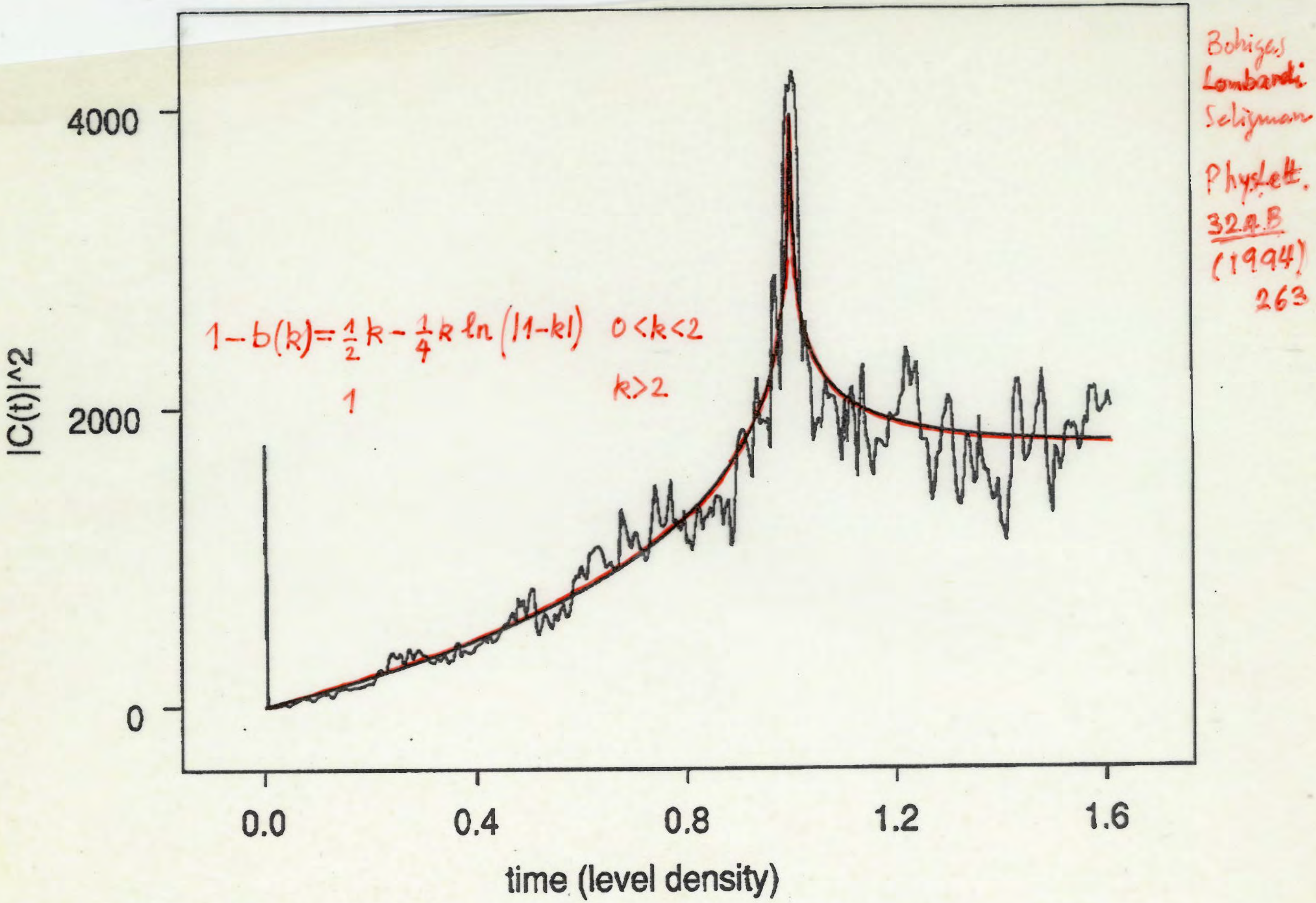
Phys Rev Lett. 54, 1645 (1985)

HIGHER-ORDER CORRELATIONS

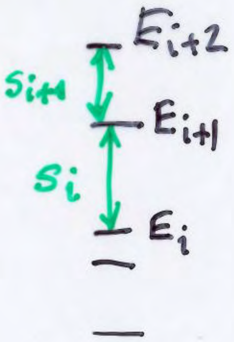
$T \downarrow$	$\hat{R}_3$ ($\times 10^2$)		$\hat{R}_4$ ($\times 10^2$)	
	NDE	GOE	NDE	GOE
0.5	0.1	0.1±0.1		
1.0	5.8	5.4±0.8	0.1	0.1±0.1
1.5	48.2	45.3±2.3	3.2	4.2±1.4
2.0	180.7	178.0±4.8	44.7	46.8±5.0

Haq, Pandey, O B
PRL 54 (1985) 1645


GOE
GSE



Nuclear Data Ensemble (1778 levels unf 3) 1 every 2 Sticks



correlation coefficient between two adjacent spacings

$$C = \sum_i (s_{i+1} - 1)(s_i - 1) \Big/ \sum_i (s_i - 1)^2$$

NDE	GOE
−0.32	−0.27 ± 0.15

Independence of widths and energy fluctuations

$$r = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{\Gamma_{ic} - \langle \Gamma \rangle}{\sigma_\Gamma} \right) \left(\frac{E_i - \langle E_i \rangle}{\sigma_E} \right)$$

NDE	GOE
0.017	0 ± 0.029

ENSEMBLES OF RANDOM MATRICES

	Wigner-Dyson	TBRE
i) spectral density in agreement with experiment	no	?
ii) spectral fluctuations in agreement with experiment	yes	yes?
iii) definition of the ensemble physically plausible	no	yes
iv) problem mathematically tractable	yes	no?

⇑ connectivity problem

random graphs

Questions:

- # of random variables

  independent

  or correlated

- choice of distribution of matrix elements

TBRE

French, Wong
Bohigas, Flores 70's

Ω : single particle states

n : fermions

2-body interaction $H = \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} \, c^+_\alpha c^+_\beta c_\delta c_\gamma$ (random variables)

The parameters are specified in the 2-particle space

Dimension N_2 of the 2-particle matrix $N_2 = \binom{\Omega}{2}$

# of independent random variables $\frac{1}{2} N_2 (N_2+1)$

Dimension N_n of the n-particle matrix $N_n = \binom{\Omega}{n}$

# of matrix elements $\frac{1}{2} N_n (N_n+1)$

expressed in terms of

For $n=2$: Wigner Dyson

Dilute limit

$2 << n << \Omega \to \infty$

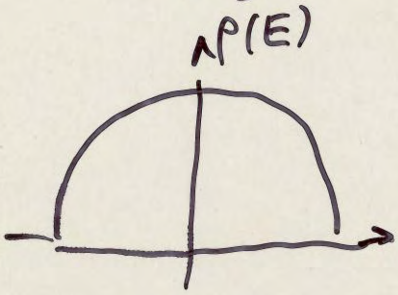
Dense limit

$n \to \infty$ $\quad \frac{n}{\Omega} \sim 1$
$\Omega \to \infty$

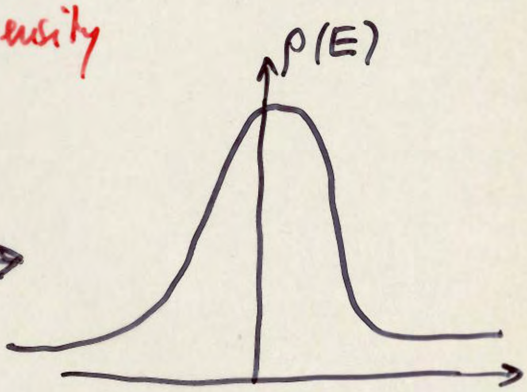
For $n=2$ $(=k)$ one recovers, by construction, Wigner-Dyson

$n = k\ (=2)$ $\qquad$ Eigenvalue density $\qquad$ $n \gg k$

WD



→



semi-circle → normal

Mon & French

Local level fluctuations seem to be WD (numerics)

TBRE

$(j_1 j_2 j_3 j_4)^6$

294x294

#23ME 1990

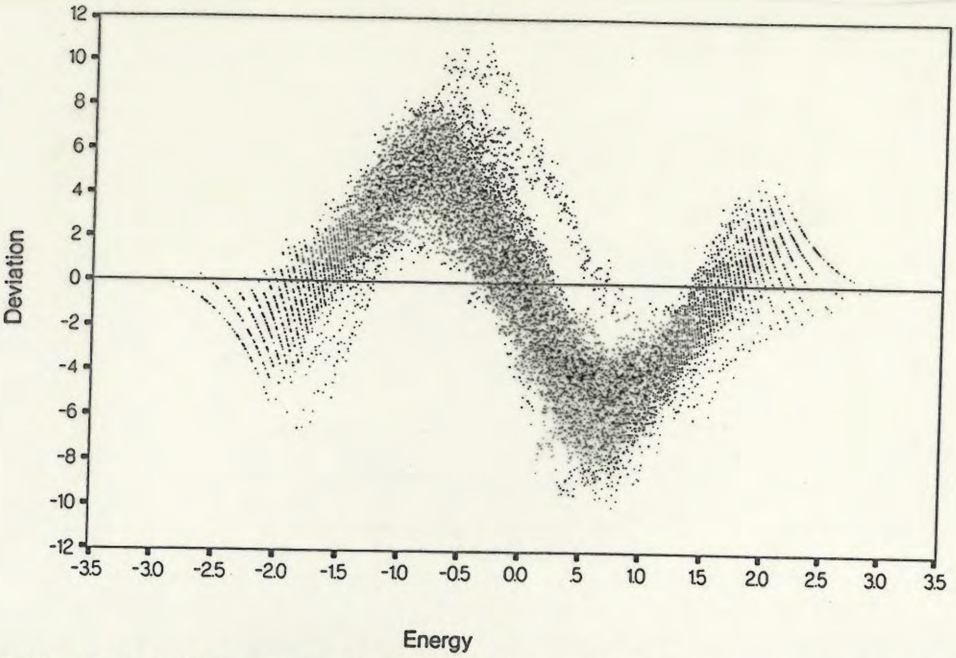


FIG. 1. Deviations $\Delta(E_i)$ vs. E_i for the TBRE. Total number of levels in the ensemble is 14 700 corresponding to 50 matrices of dimensionality 294. the smooth density $\rho_s(E)$ was taken as a Gaussian.

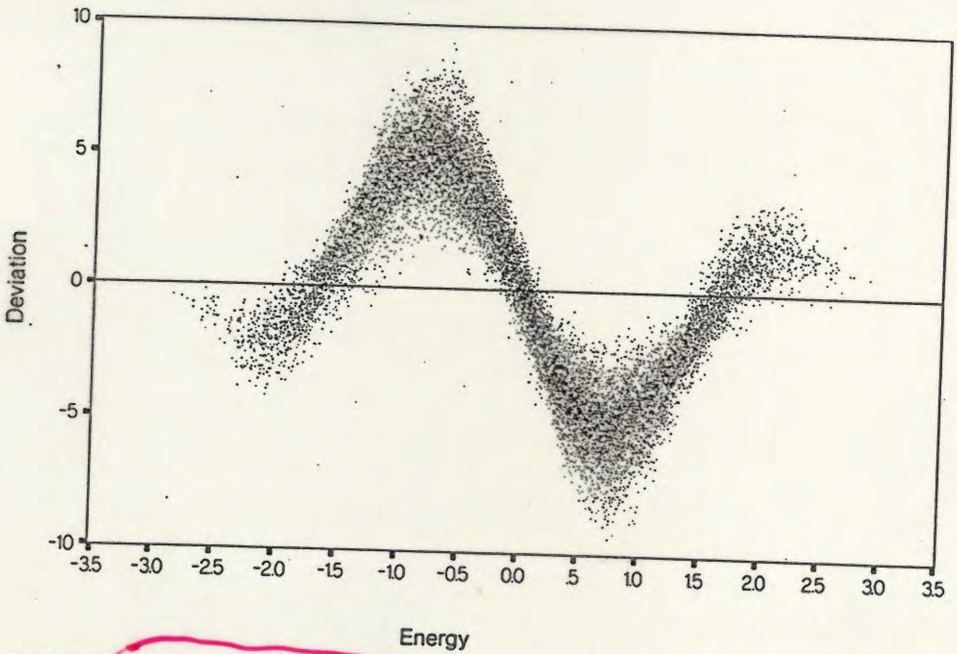


FIG. 2. The deviations for order 3 of the GC expansion with optimized constants for the TBRE of dimensionality 14 700.

Laberge
Haq
Can. J. Phys.
68 (1990) 301

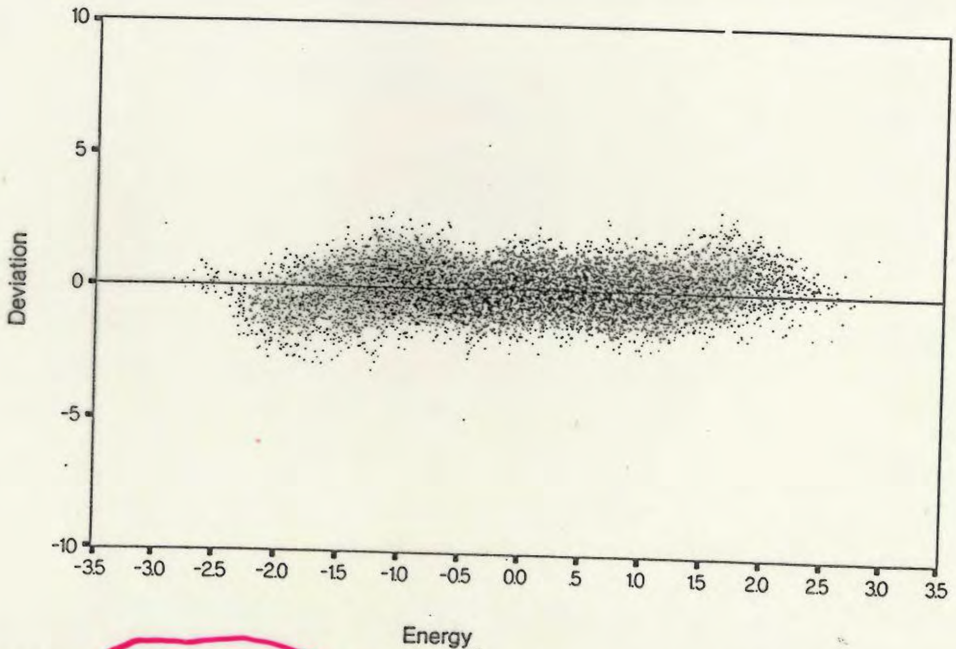
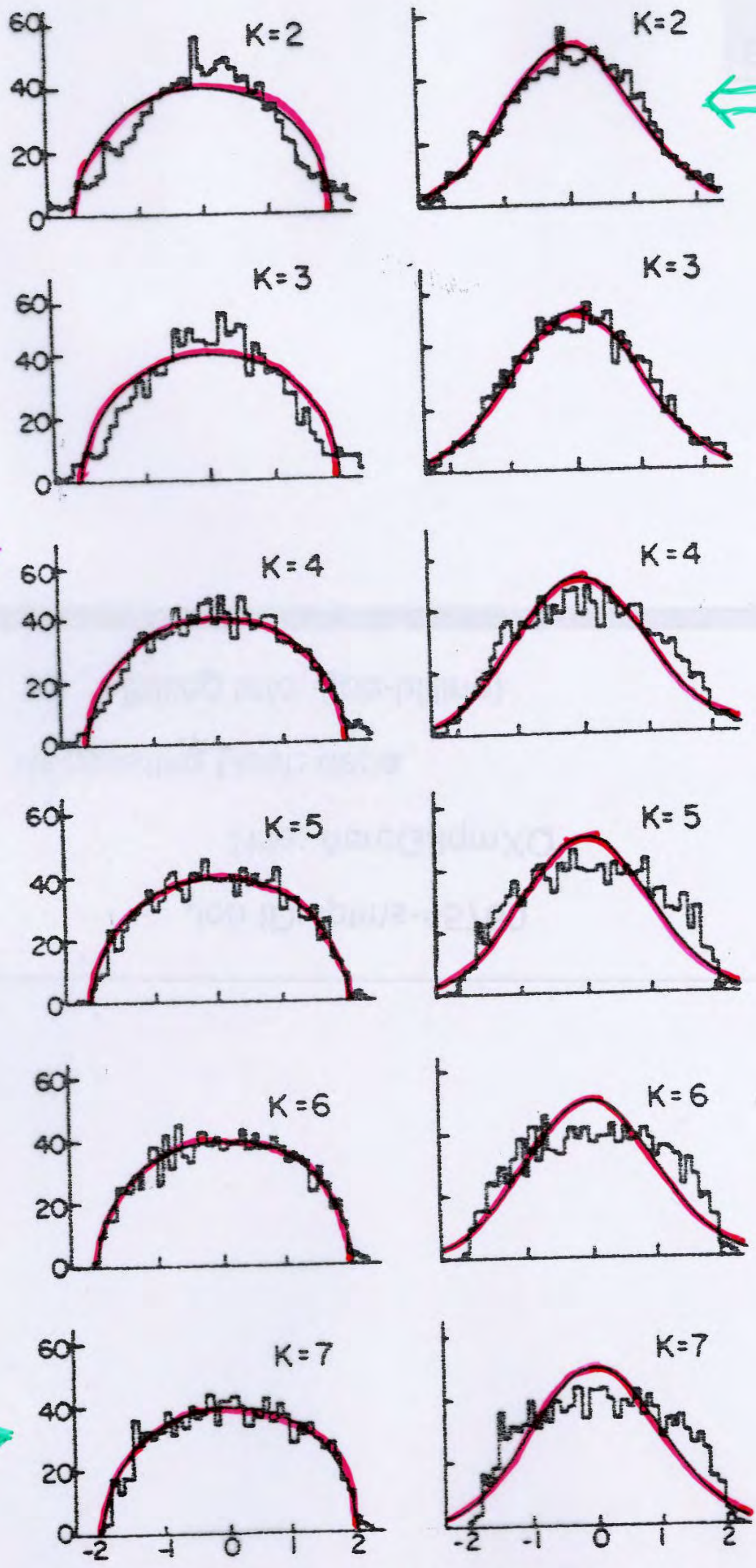


FIG. 3. The deviations for order 4 of the GC expansion with optimized constants for the TBRE of dimensionality 14 700

Compared to SC | Compared to Gaussian | # independent 2BME



J. B. French

SSM Wong

Phys. Lett. 35B (1971) 5

2356

f^7 ($J=7/2$) ensemble (25×25)-matrices for k-body ($k=2,\dots 7$)

Phys. Rev. Lett. 87 (2001) 010601

Benet, Rupp, Weidenmüller Ann. Phys. (NY) 292 (2001) 67

k, n, Ω

Spectral fluctuations

$k \leq n < 2k$ Wigner-Dyson

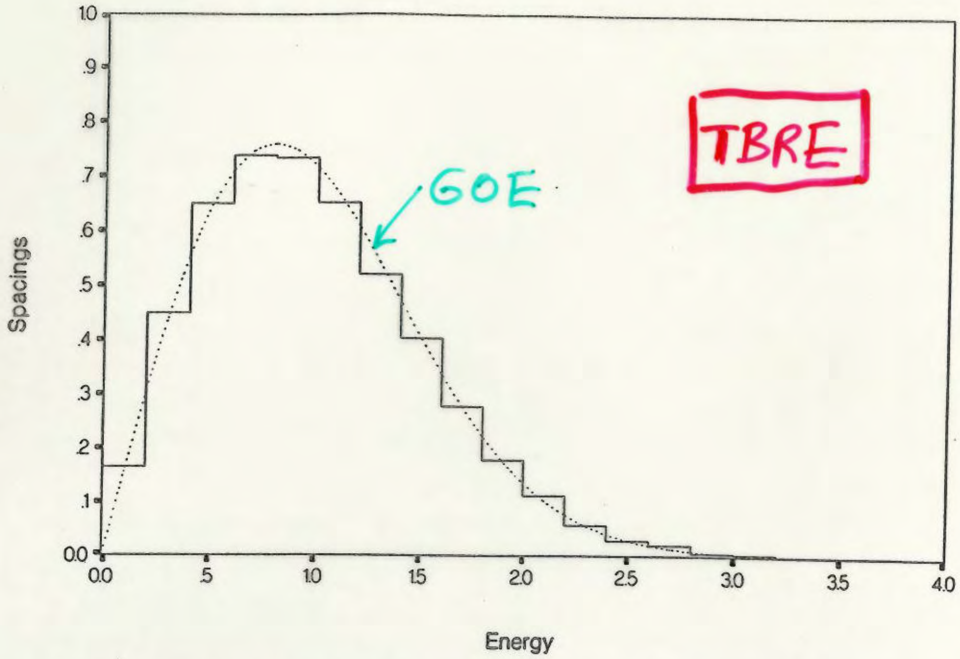
$k << n << \Omega$ Poissonian!

Agreement between Wigner Dyson and nuclear resonance data would be a finite size effect ?

M. Srednicki, cond-mat/0207201

L. Benet, H. A. Weidenmüller, cond-mat/0207656
J. Phys. A36 (2003) 3569

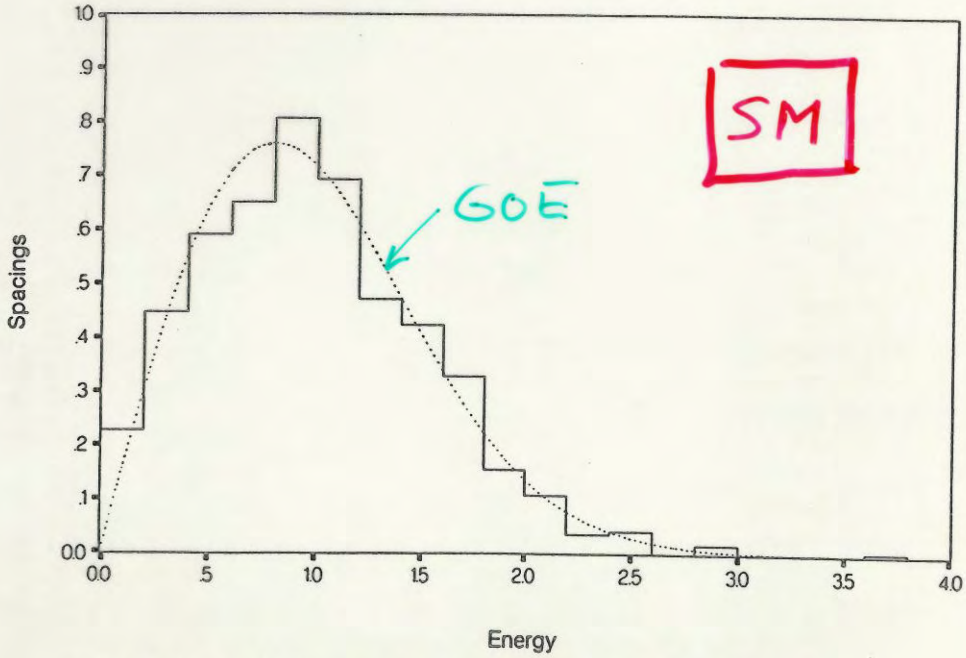
The spectral statistics of the k-body random-interaction model remains an open question



$(j_1, j_2, j_3, j_4)^6$

FIG. 5. Nearest-neighbour spacing distribution for the TBRE using 4th order unfolding.

$(ds)^{12}$



$(d, s)^{12}$

FIG. 6. Nearest-neighbour spacing distribution for SM of dimensionality 839.

Laberge, Haq, Can. J. Phys. 68 (1990) 301

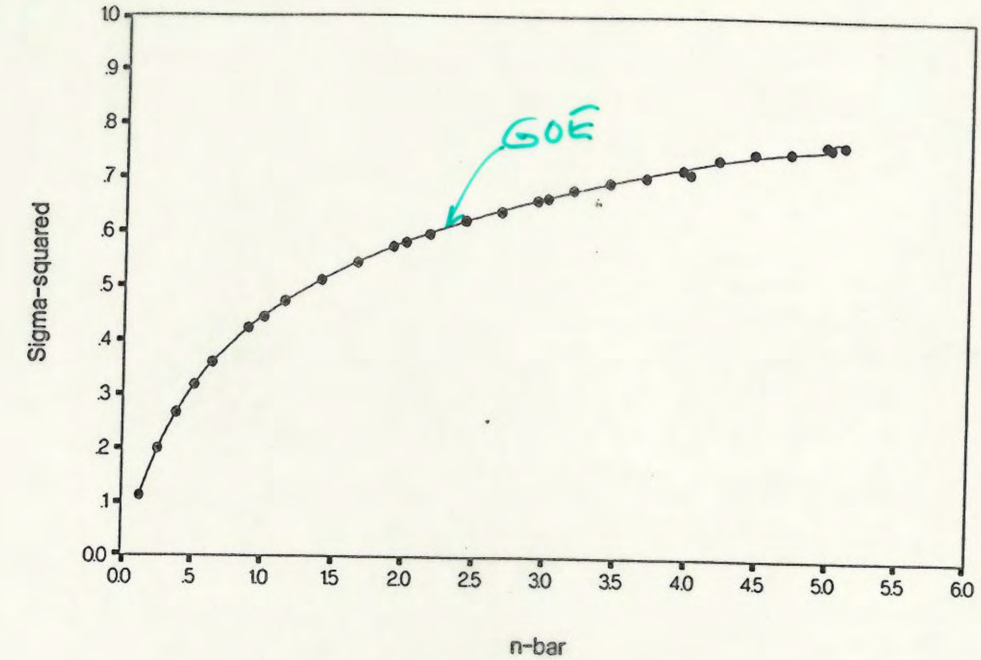


FIG. 8. Number variance $\sum^2(\bar{n})$ for TBRE.

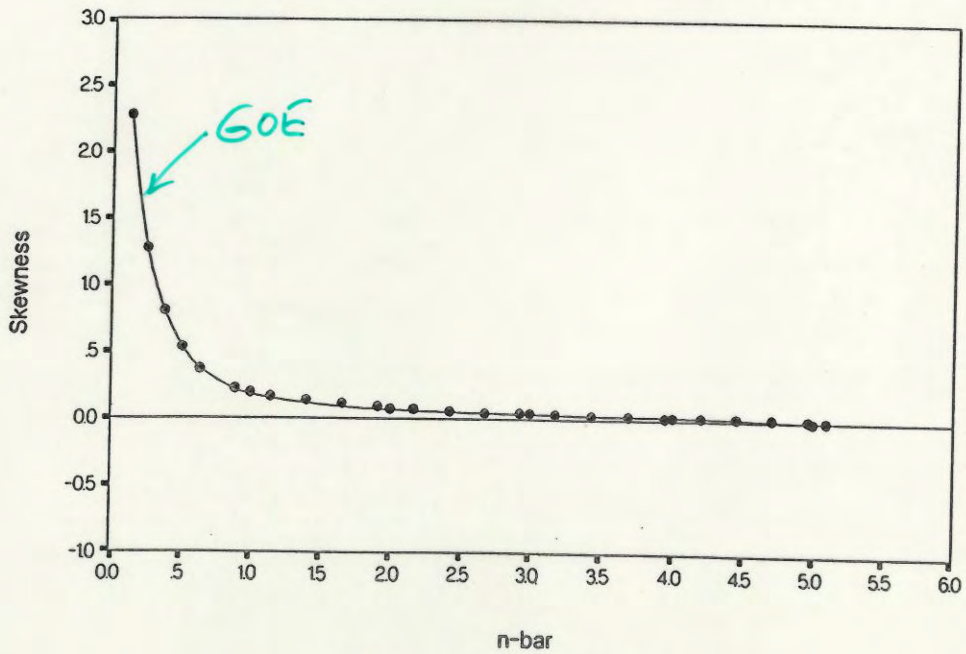


FIG. 9. Skewness $\gamma_1(\bar{n})$ for TBRE.

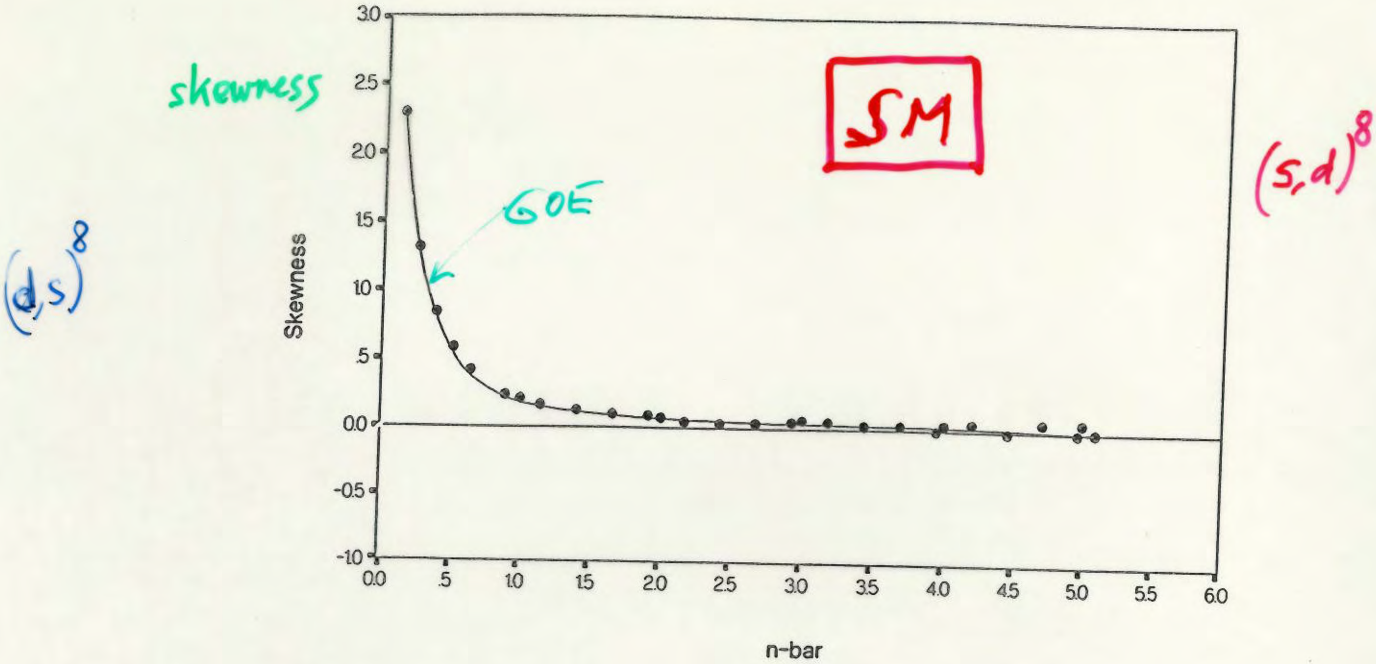


FIG. 10. Skewness $\gamma_1(\bar{n})$ for SM(1206).

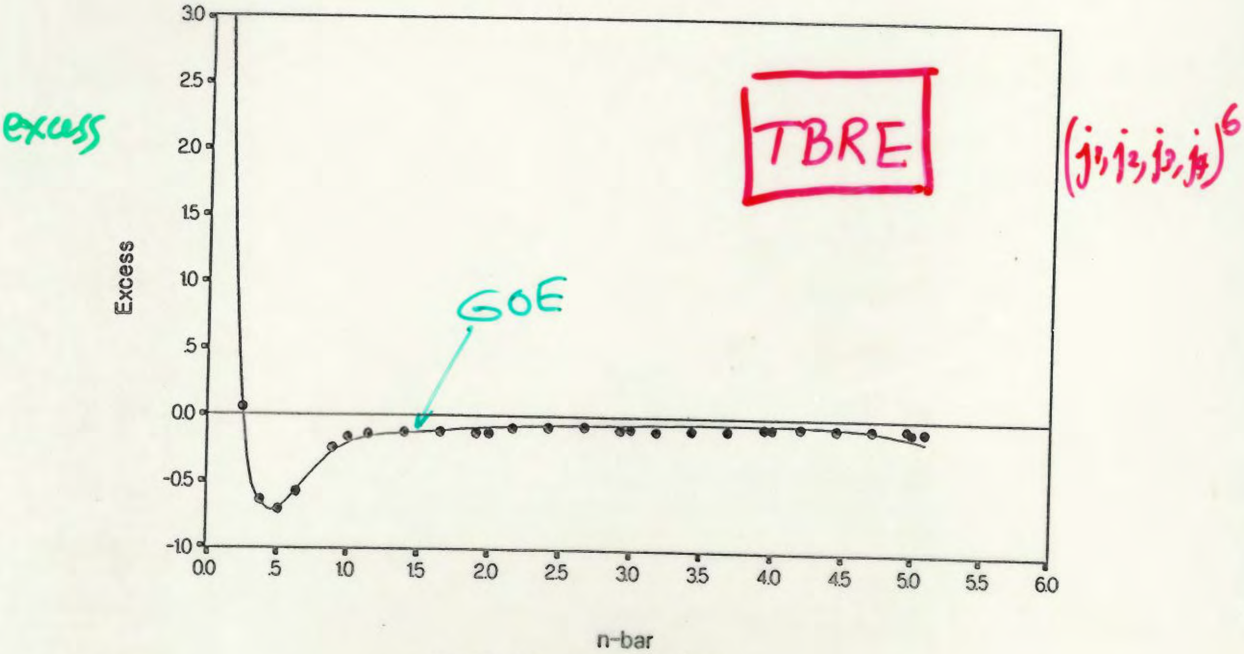


FIG. 11. Excess $\gamma_2(\bar{n})$ for TBRE.

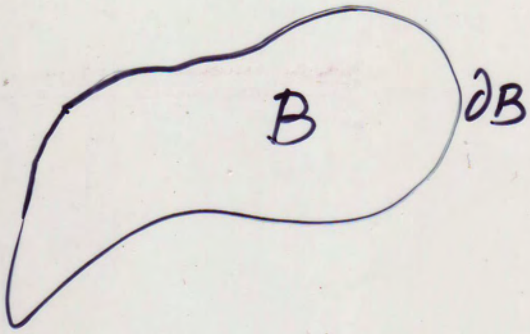
Can. J. Phys. 68 (1990) 301

Clàssic

billar

Quàntic

partícula lliure en una capsa



Modes propis de la capsa: solucions de

equació de Schrödinger (o de Helmholtz) $(\Delta + E)\Psi = 0$

$\psi\big|_{frontera} = 0$ Dirichlet

Espectre del laplacià

$\frac{d\psi}{dn}\Big|_{frontera} = 0$ Neuman

$N(E)$: funció de comptatge (escala)

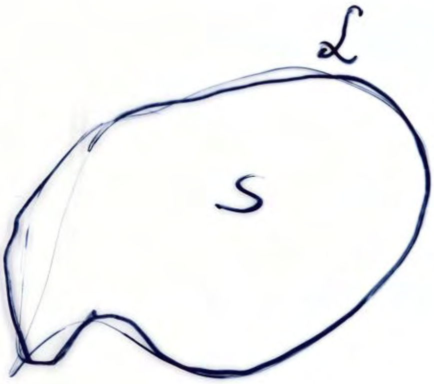
Weyl $\bar{N}(E) = \frac{1}{4\pi}\left(S E \mp \alpha\sqrt{E} + K + \dots\right)$

(superfície → S; perímetre → α; integral de curvatura → K)

M. Kać: 'Can one hear the shape of a drum'?

Separation of average and fluctuations.

$$N(E) = N_{av}(E) + N_{fl}(E)$$
$$= \langle N(E) \rangle + N_{fl}(E)$$



Vibrating membrane

$$(\Delta + E)\psi = 0$$

with Dirichlet $\psi\big|_{\text{boundary}} = 0$

Weyl

$$N_{av}(E) = \frac{1}{4\pi}\left(SE - \mathcal{L}\sqrt{E} + K\right)$$

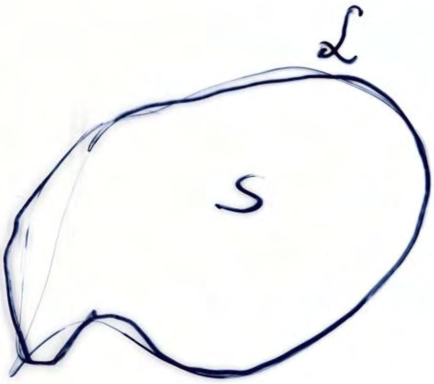
one particle in a potential $V(\underset{\sim}{r})$

$$(\Delta - V(\underset{\sim}{r}) + E)\psi = 0$$

$$N_{av}(E) = \frac{1}{2^d \pi^{\frac{d}{2}} \Gamma\left(\frac{d}{2}+1\right)} \int_{V(\underset{\sim}{r}) \leq E} \left\{ E - V(\underset{\sim}{r}) \right\}^{\frac{d}{2}} d\underset{\sim}{r}$$

Separation of average and fluctuations.

$$N(E) = N_{av}(E) + N_{fl}(E)$$
$$= \langle N(E) \rangle + N_{fl}(E)$$



Vibrating membrane

$$(\Delta + E)\psi = 0$$

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Weyl

$$N_{av}(E) = \frac{1}{4\pi}\left(SE - \mathcal{L}\sqrt{E} + K\right)$$

one particle in a potential $V(\underset{\sim}{r})$

$$(\Delta - V(\underset{\sim}{r}) + E)\psi = 0$$

$$N_{av}(E) = \frac{1}{2^d \pi^{\frac{d}{2}} \Gamma\left(\frac{d}{2}+1\right)} \int_{V(\underset{\sim}{r}) \leq E} \left\{ E - V(\underset{\sim}{r}) \right\}^{\frac{d}{2}} d\underset{\sim}{r}$$

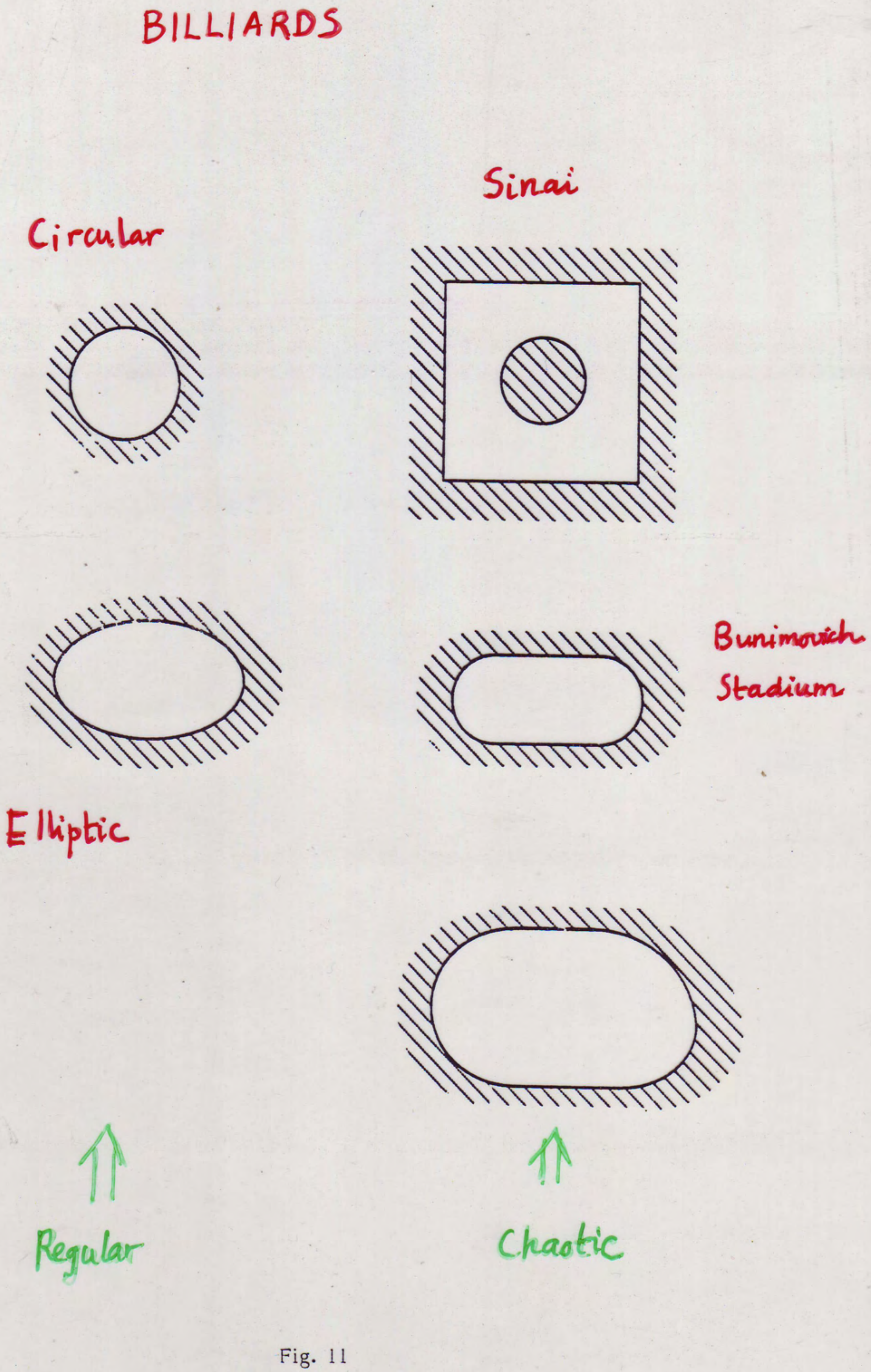
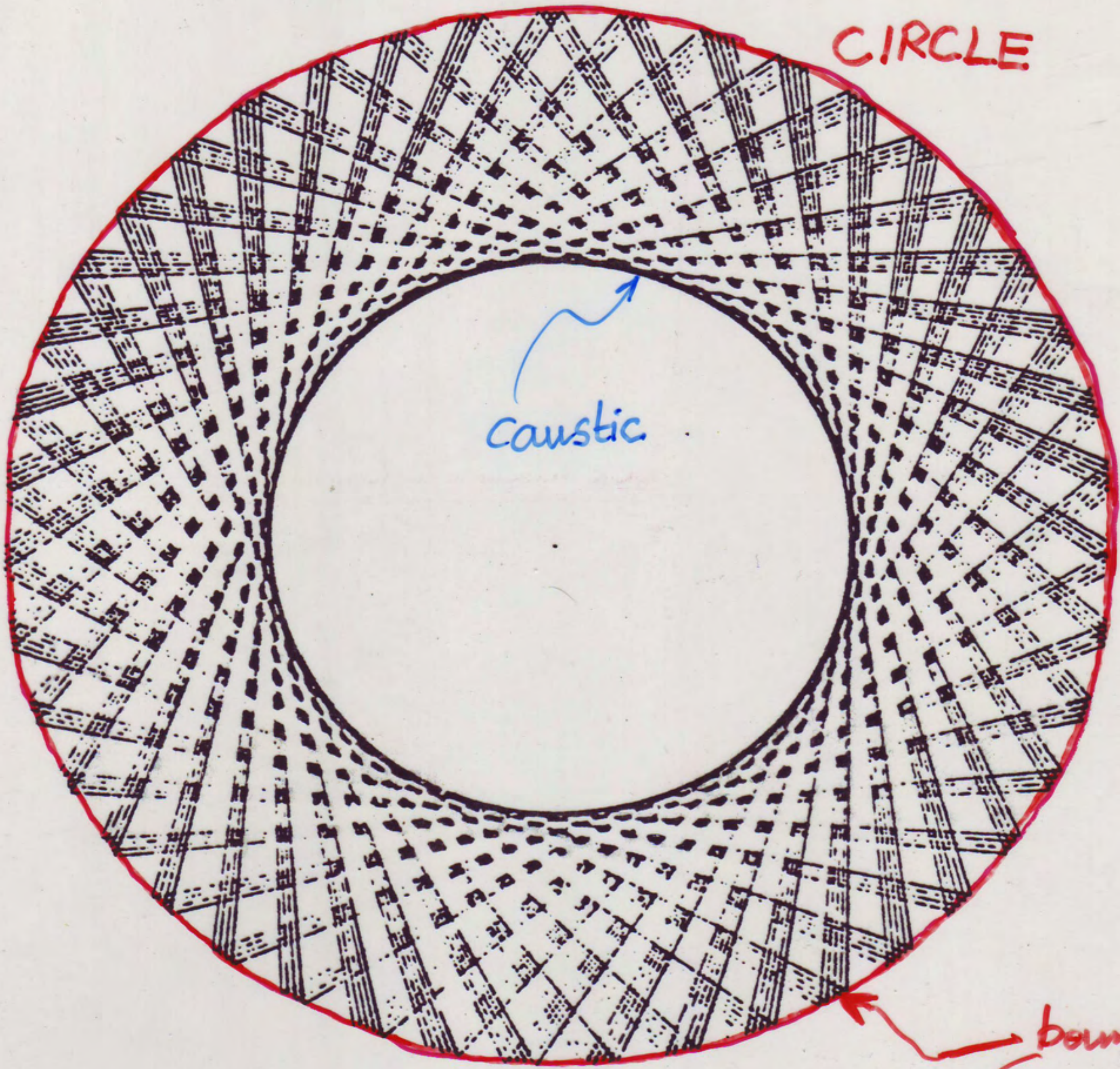
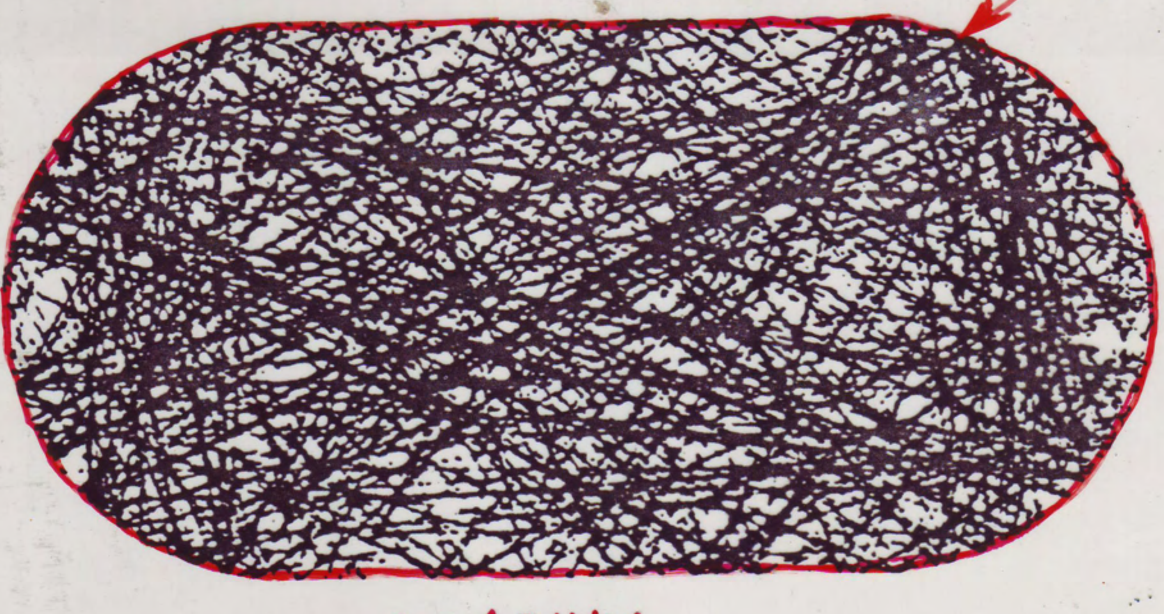

BILLIARDS
Sinai
Circular
Bunimovich
Stadium
Elliptic
Regular
Chaotic

Fig. 11

BILLIARDS





STADIUM

Chaotic system

Sinai billiard

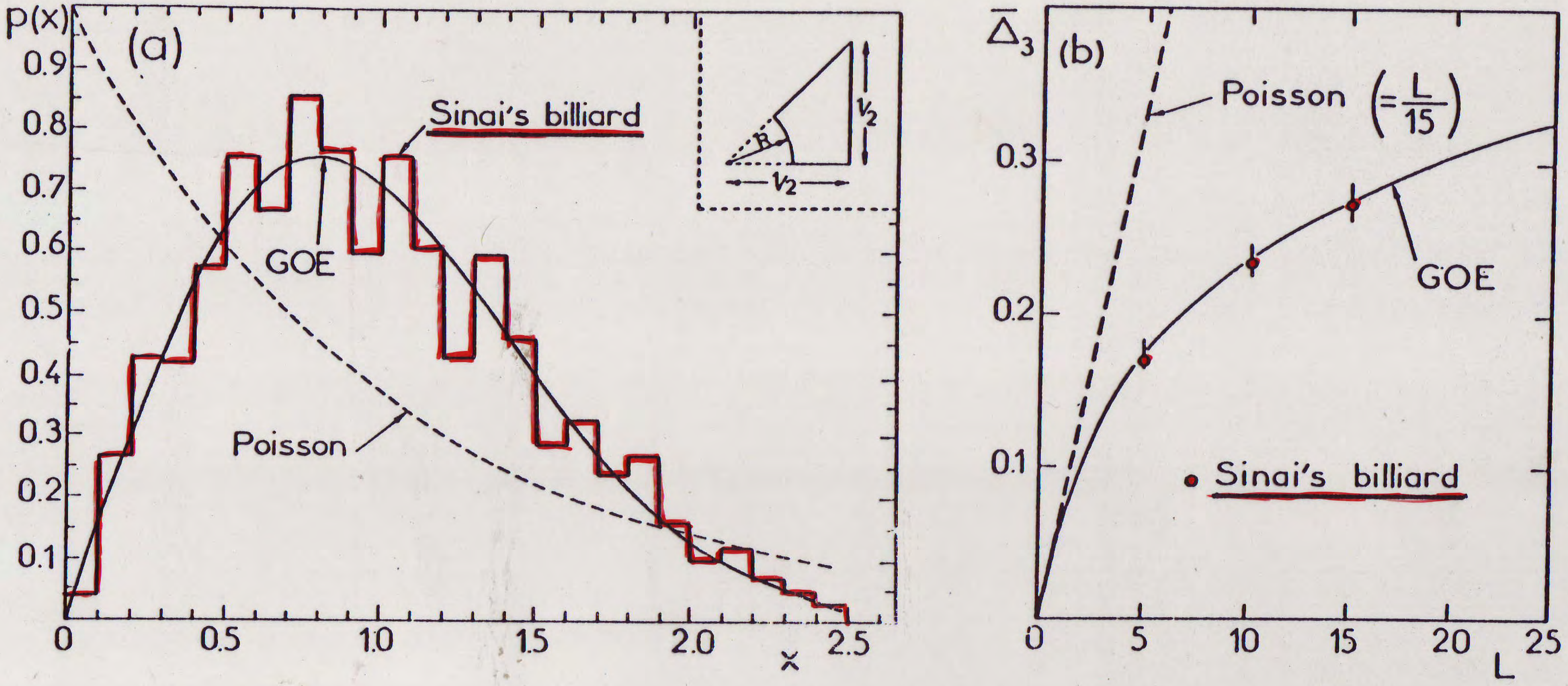


Fig. 12

Bohigas, Giannoni & Schmit
Phys. Rev. Lett. 52 (1984) 1

Lagrange's equations.

The state of the system is completely characterized by the set of all (generalized) coordinates and (generalized) velocities

$$q_i \,, \; \dot{q}_i \qquad i = 1, 2, \ldots, d$$

To a system there corresponds a Lagrangian

$$L = L(q_1, \ldots, q_d, \dot{q}_1, \ldots, \dot{q}_d, t)$$

Hamilton's principle minimizing the action integral

$$\int_{t_1}^{t_2} L(\underset{\sim}{q}, \dot{\underset{\sim}{q}}, t)\, dt$$

leads to Lagrange's equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0} \qquad i = 1, 2, \ldots, d \,,$$

a set of d second order equations.

Examples.

System of non interacting particles

$$L = T = \sum_i \frac{1}{2} m_i \dot{q}_i^2$$

If there is a potential

$$L = T - V = \sum_i \frac{1}{2} m_i \dot{q}_i^2 - V(q_1, \dots, q_d)$$

Force F_i acting on each particle

$$F_i = -\frac{\partial}{\partial q_i} V(q_1, \dots, q_d)$$

□

$$m_i \ddot{q}_i = F_i \qquad \text{(Newton's equations)}$$

Properties.

If $\frac{\partial L}{\partial t} = 0$ (closed system)

one has the "energy E" is constant

$$E = \sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = T + V$$

Hamilton picture.

Definition of generalized momenta p_i

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad i = 1, 2, \ldots, d$$

(for cartesian coordinates $p_i = m_i \dot{q}_i$)

$$H(p, q, t) = \sum p_i \dot{q}_i - L(q, \dot{q}, t)$$

Provided that

$$\det \left| \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right| \neq 0$$

$$p_i(q, \dot{q}, t) = \frac{\partial L}{\partial \dot{q}_i}(q, \dot{q}, t)$$

can be inverted to express the $\dot{q}_i$ in terms of the p_i

Example

$$L = \sum \frac{1}{2} m_i \dot{q}_i^2 - V(q_1, \ldots, q_d)$$

$$p_i = m_i \dot{q}_i \qquad \dot{q}_i = \frac{p_i}{m_i}$$

$$H(p, q) = \sum \frac{p_i^2}{2m_i} + V(q_1, \ldots, q_d)$$

$$H(p,q) = \sum p_i \dot{q}_i - L(q,\dot{q})$$

$$dH = \sum_i \left(p_i\, d\dot{q}_i + \dot{q}_i\, dp_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right)$$

$\frac{\partial L}{\partial q_i}$ → Lagrange → $\dot{p}_i$; $\frac{\partial L}{\partial \dot{q}_i} = p_i$

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \end{cases}$$

Hamilton's equations

set of d first order differential equations

Properties.

$$\sum \frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} = 0$$

Liouville's theorem: a blob of phase-space fluid is incompressible

Volume element in phase space is preserved under the Hamiltonian flow

A: region of phase space

$\mu(A)$: Liouville's measure

$$\mu(A) = \int_A dq_1 \ldots dq_d\, dp_1 \ldots dp_d$$

$$\mu(T_t A) = \mu(A)$$

CLASSICAL MECHANICS

Conservative Hamiltonians with d degrees of freedom

i) for a given set of initial conditions, the dimensionality of the accessible surface in phase space is $\leq 2d-1$; the energy is constant along the energy surface

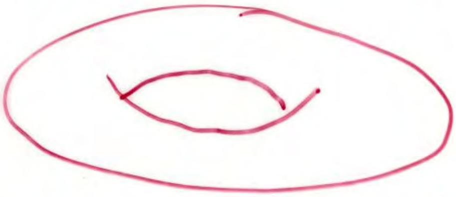
ii) the volume element in phase space is preserved (Liouville); the Hamiltonian flow is incompressible

iii) trajectories in phase space cannot cross

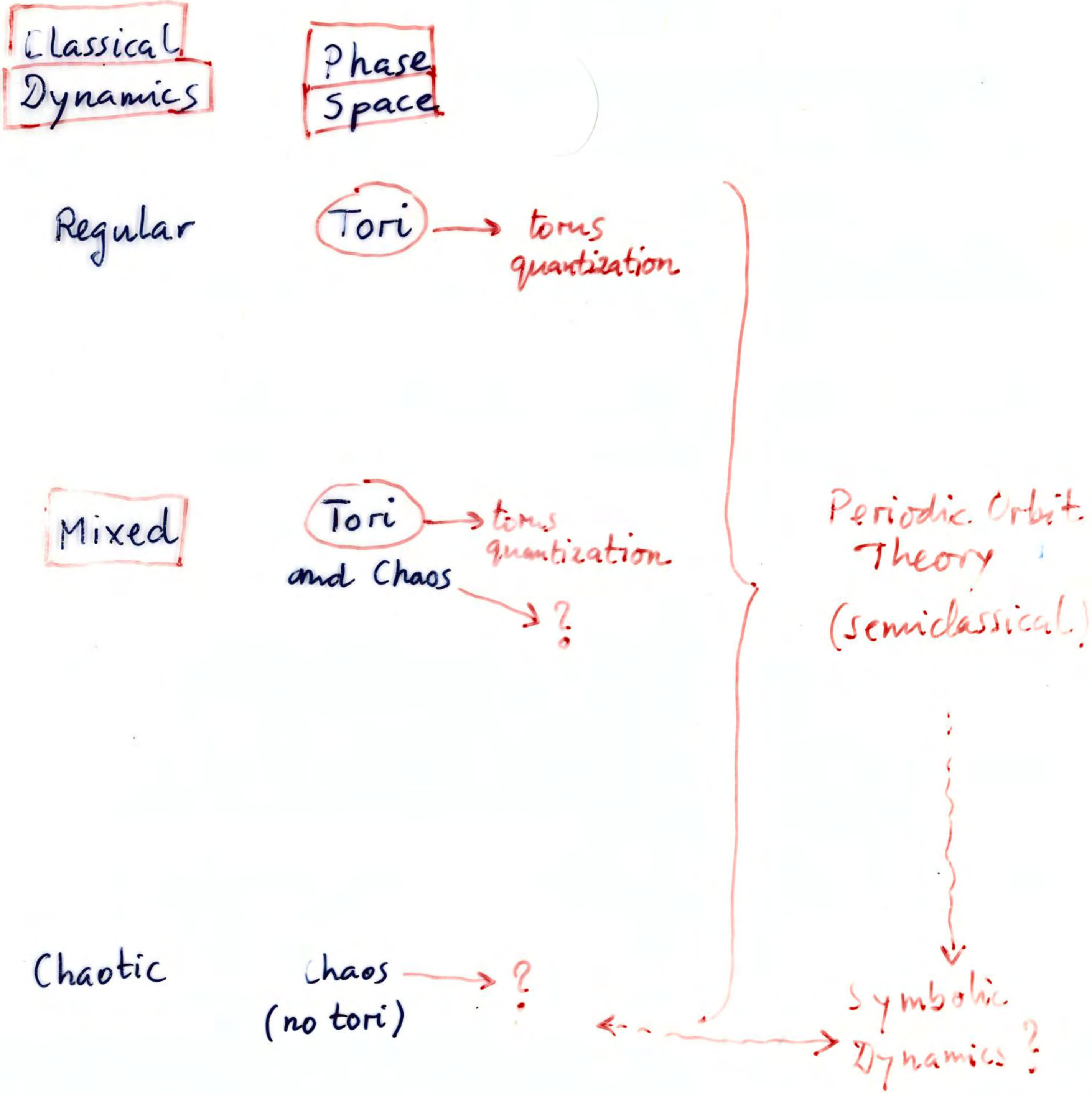
CLASSICAL MECHANICS

Regular (integrable) motion

For each set of initial conditions, the accessible surface is a compact manifold having the topology of a d-dimensional torus in phase space (d degrees of freedom)



Each value of $\underset{\sim}{I}$ defines the torus, whereas the vector $\underset{\sim}{\vartheta}(t)$ gives the position of the trajectory on the torus at each time t.


Classical Dynamics
Phase Space
Regular
Tori
torus quantization
Mixed
Tori and Chaos
torus quantization
?
Chaotic
Chaos (no tori)
?
Periodic Orbit Theory (semiclassical)
Symbolic Dynamics ?

CLASSICAL MECHANICS

The route from regular (integrable) to irregular (chaotic) motion

Regular

- Integrable system
- Ergodic : Covering uniformly the energy surface
- Mixing : Asymptotic equilibration
- K-system : Exponential instability

↓

Irregular

CLASSICAL MECHANICS — QUANTUM MECHANICS

d degrees of freedom

Limit — INTEGRABLE (regular)

d constants of the motion

Energy shell : d-dimensional torus

correspondence ⟷ principle

O.K.

MIXED (generic case)

Coexistence of tori and chaotic regions

Correspondence ⟷ Principle

O.K for tori

for chaotic regions **

CHAOTIC (irregular) — Limit

Energy shell : dimension 2d-1

Generic trajectory visits all available phase space

**

Laplace, 1776

"The present state of the system of nature is evidently a consequence of what it was in the preceding moment, and if we conceive of an intelligence which at a given instant comprehends all the relations of the entities of this universe, it could state the respective positions, motions, and general affects of all these entities at any time in the past or future.

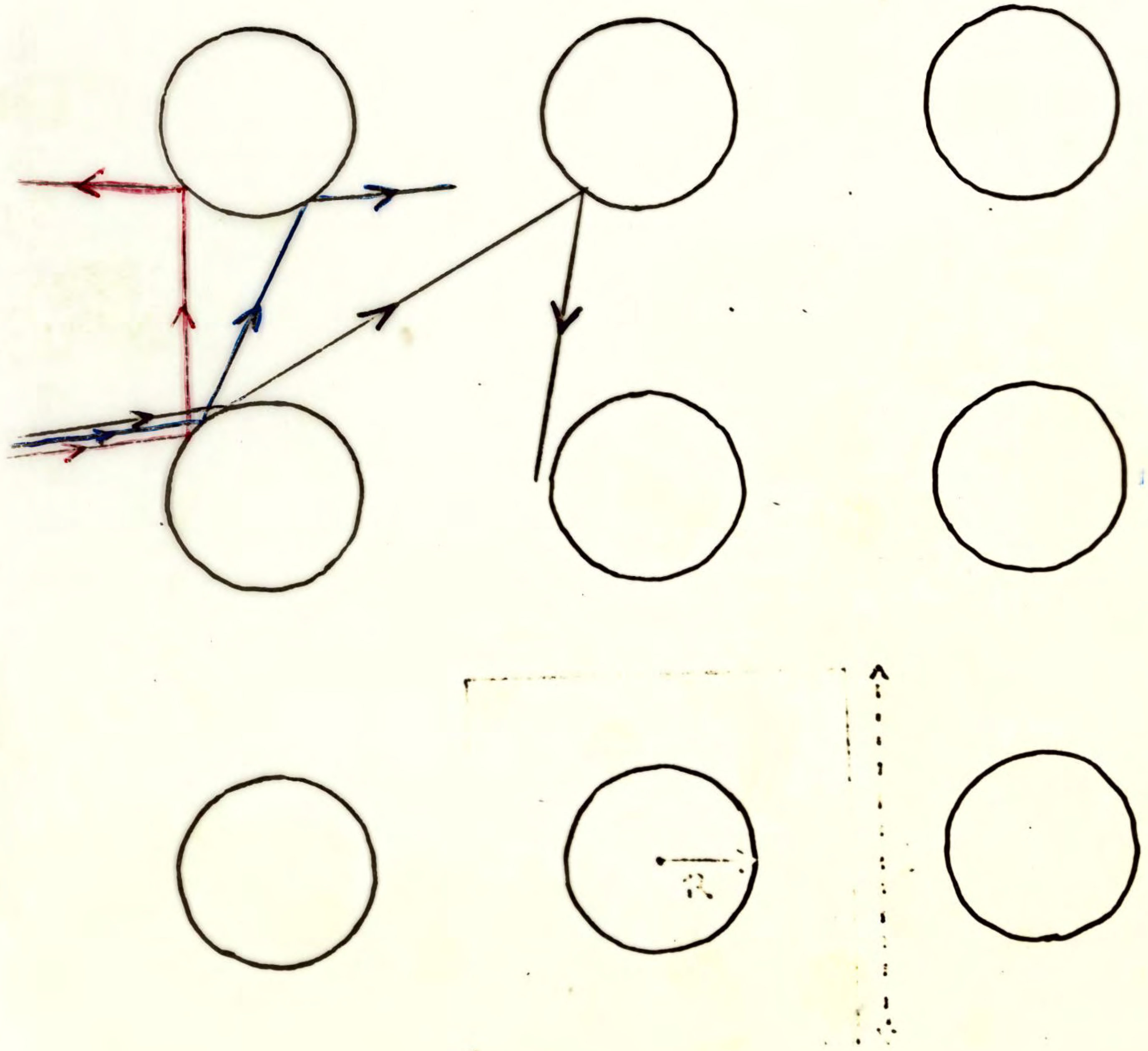
"Physical astronomy, the branch of knowledge which does the greatest honor to the human mind, gives us an idea, albeit imperfect, of what such an intelligence would be. The simplicity of the law by which the celestial bodies move, and the relations of their masses and distances, permit analysis to follow their motions up to a certain point; and in order to determine the state of the system of these great bodies in past or future centuries, it suffices for the mathematician that their position and their velocity be given by observation for any moment in time. Man owes that advantage to the power of the instrument he employs, and to the small number of relations that it embraces in its calculations. But ignorance of the different causes involved in the production of events, as well as their complexity, taken together with the imperfection of analysis, prevents our reaching the same certainty about the vast majority of phenomena. Thus there are things that are uncertain for us, things more or less probable, and we seek to compensate for the impossibility of knowing them by determining their different degrees of likelihood. So it is that we owe to the weakness of the human mind one of the most delicate and ingenious of mathematical theories, the science of chance or probability."

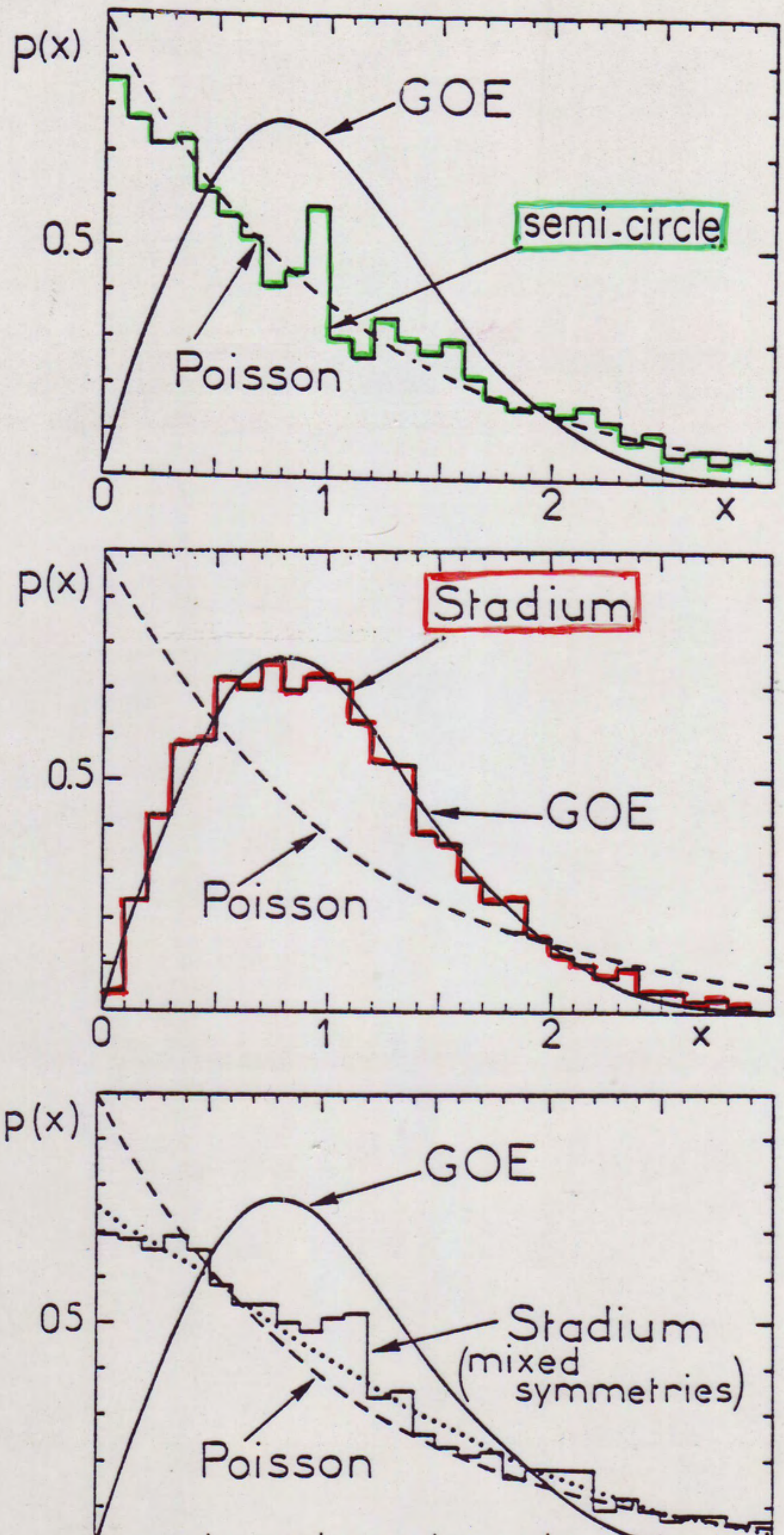
Poincaré, 1903

"A very small cause which escapes our notice determines a considerable effect that we cannot fail to see, and then we say that the effect is due to chance. If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation *approximately*. If that enabled us to predict the succeeding situation with *the same approximation*, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon."

2- Systèmes chaotiques

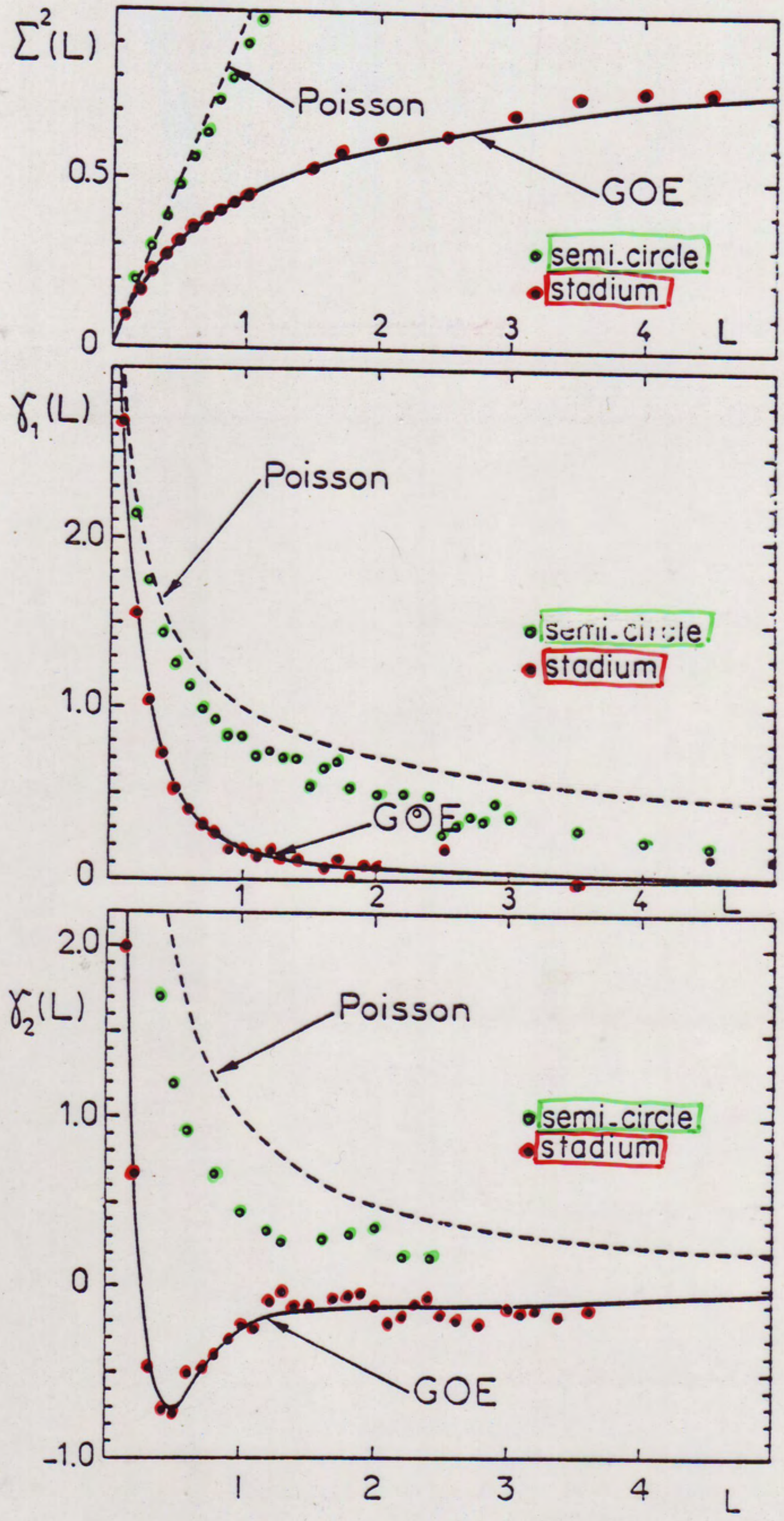
BILLARD DE SINAI





Bohigas, Giannoni & Schmit
J. Physique Lett. 45 (1984) L-1015

Fig. 13



Bohigas, Giannoni, Schmit
J. Physique Lett. 45 (1984) L-1015

Fig. 14

Hermitian random matrices can serve as models of Hamiltonians of chaotic autonomous systems (cf GOE, GUE, GSE).

Unitary random matrices (COE, CUE, CSE) play an analogous role for chaotic periodically driven systems, as models of the unitary evolution operator.

BGS or random matrix conjecture

In contrast, spectral fluctuations of regular systems are Poissonian (Berry & Tabor).
Sinai

(iii)

TO THE

READER.

THE Irregularity of the Moon's Motion hath been all along the just Complaint of Astronomers; and indeed I have always look'd upon it as a great Misfortune that a Planet so near us as the Moon is, and which might be so wonderfully use-

A 2 *ful*

iv To the Reader.

ful to us by her Motion, *as well as her* Light *and* Attraction *(by which our Tides are chiefly occasioned) should have her Orbit so unaccountably various, that it is in a manner vain to depend on any Calculation of an* Eclipse, *a* Transit, *or an* Appulse *of her, tho never so accurately made. Whereas could her Place be but truly calculated, the Longitudes of Places would be found every where at Land with great Facility, and might be nearly guess'd at Sea without the help of a Telescope, which cannot there be used.*

This Irregularity of the Moon's Motion depends (as is now well known, since Mr. Newton *hath demonstrated the Law of Universal Gravitation) on the Attraction of*

I. Newton, "A new and most Accurate Theory of the Moons Motion; Written that Incomparable Mathematician Mr. Isaac Newton" London, Baldwin, 1702

3-body Sun - Earth - Moon

Poincaré : partir des solutions périodiques (les corps mobiles décrivent des courbes fermées);

étudier ce qu'il se passe au voisinage d'une trajectoire périodique

' (l'étude des trajectoires périodiques) est la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable.'

A Scandinavian episode (before NORDITA!)

Oscar II, King of Sweden. Graduated from Uppsala

Gösta Mittag-Leffler: graduated also from Uppsala (1872)

Went as a 'postdoc' to Paris (Hermite) and Berlin (Weierstraß).

Came back to Sweden and became professor at Stockholm.

Convinced Oscar II to publish Acta Mathematica and later to create a prize

subjects selected by Hermite and Weierstraß, (joined by Mittag-Leffler as jury members)

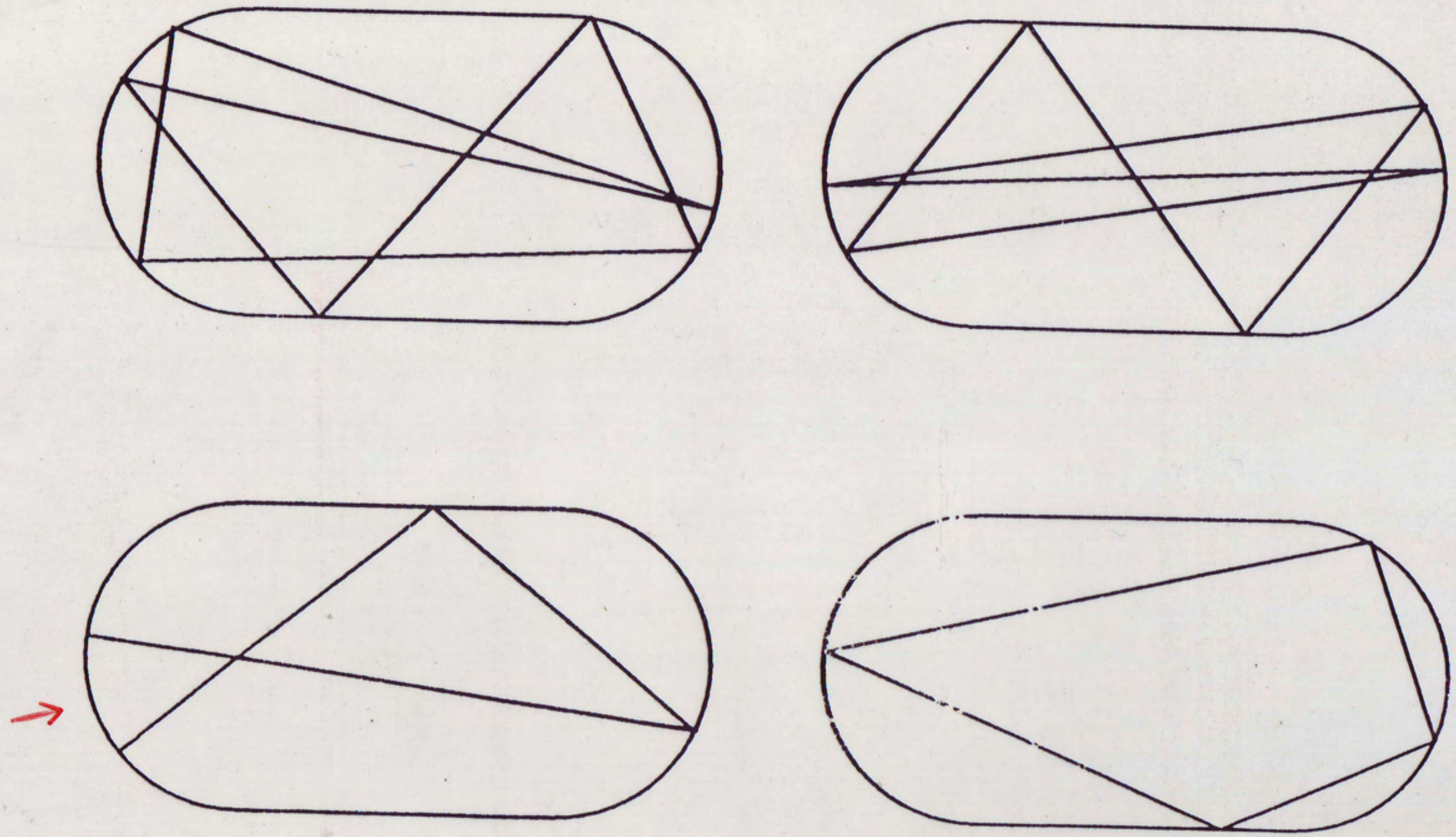
Deadline for submission: June 1888

Should be awarded January 89, Oscar's II birthday

'Sur le problème à trois corps et les équations de la dynamique'.

Edward Phragmén, July 89

Final version: end of 1890



S. Tomsovic

The density of states may be computed from the Green's function

$$\rho(E) = -\frac{1}{\pi}\int d\underset{\sim}{q} \lim_{\epsilon\to 0^+} G(\underset{\sim}{q}, \underset{\sim}{q}, E+i\epsilon)$$

$G(q, q, E)$ is the propagator for paths of energy E that start at $\underset{\sim}{q}$ and come back to $\underset{\sim}{q}$. In the limit $\hbar\to 0$, the leading contribution to G is a sum over all classical trajectories that start and come back to $\underset{\sim}{q}$. A stationary phase approximation of the integral over $\underset{\sim}{q}$ selects, among all the classical closed trajectories, those that start and come back to a given point with the same momentum (periodic trajectories).

Periodic orbit theory. cf. Gutzwiller, Bloch & Balian

$$\rho(E) = \sum \delta(E - E_i)$$

Based on the saddle-point approximation to Feynman's path integral for $\hbar \to 0$

$$\rho(E) = \langle \rho(E) \rangle + \rho_{fe}(E)$$

$$\rho_{fe}(E) \simeq \sum_{p.o} A e^{iS}$$

$$\sum \delta(E - E_i) \simeq \sum_{p.o.} A e^{iS}$$

↑ Quantum

↑ Classical information

Interval ΔE	energy scales	Corresponds to microscope	time scales	# of levels it contains
(1) h^d	mean spacing δ	h^{-d}	Heisenberg time $\tau_H = \frac{h}{\delta}$	1
(2) h	h/τ_{min}	h^{-1}	period of shortest periodic orbit τ_{min}	$h^{-(d-1)}$

h : Planck's constant

d : # of freedoms

(1) Inner scale : Dominated by properties of extremely long orbits

(2) Outer scale : The orbits which are not long differ from system to system and break down universalities at large scale. The outer scale is classically small but semiclassically large.

Proc. Roy. Soc. London A400 (1985) 229

M.V. Berry

SEMICLASSICAL THEORY

Gutzwiller
Balian & Bloch
Berry

$$\rho(E) = \sum_i \delta(E - E_i)$$

smooth — oscillating

$$\rho(E) = \bar{\rho}(E) + \tilde{\rho}(E)$$

Gutzwiller trace formula

$$\tilde{\rho}(E) = 2 \sum_{p} \sum_{r=1}^{\infty} A_{p,r}(E) \cos\left[r S_p(E)/\hbar + \nu_{p,r} \right]$$

$\tilde{\rho}(E)$: quantum mechanics

p: primitive periodic orbits

r: repetitions

$A_{p,r}$, S_p, $\nu_{p,r}$: classical mechanics

S_p action ; $\tau_p = dS_p/dE$ period of periodic orbit p

$\nu_{p,r}$ Maslov index

$$A_{p,r} = \frac{\tau_p}{h\sqrt{|\det[M_p^r - I]|}}$$

Chaotic systems

$$A_p^2 = \frac{(2\pi)^{d-1}}{\hbar^{(d+1)} \tau_p^d \left| \det\{\partial\omega_j/\partial I_k\}_p \sum_j \omega_j \cdot \frac{\partial I_j}{\partial \tau_p} \right|}$$

Integrable systems

d : # degrees of freedom

h : Planck's constant

(I, ω) : action-angle variables

frequencies $\omega_j = \frac{\partial H}{\partial I_j} = 2\pi \frac{m_j}{\tau_p}$

m_j : integers

Integrable systems

action I_j, angle ϑ_j variables

caracteristic frequencies $\omega_j = \frac{\partial H}{\partial I_j}$

quantization

commensurability of frequencies

$$\omega_j (I_1, \ldots, I_d) = 2\pi \, m_j / \tau$$

m_j : integers

$$A_p^2 (E) = \frac{(2\pi)^{d-1}}{\hbar^{d+1} \tau_p^d \left| \det \left\{ \frac{\partial \omega_i}{\partial I_k} \right\}_p \sum_j \omega_j \cdot \frac{\partial I_j}{\partial \tau_p} \right|}$$

For **chaotic systems**

$$A_{p,r}(E) = \frac{\tau_p}{2\pi\hbar\sqrt{\left|\det\left(M_p^r - I\right)\right|}}$$

M_p : monodromy matrix

governs the motion in the vicinity of the periodic motion. Contains the information on the unstability

When there is a Lyapunov exponent much larger than the others

$$A_{p,r}(E) \approx \frac{\tau_p}{2\pi\hbar} \exp\left(-\frac{r}{2}\lambda_p^{max}\right)$$

M. V. Berry : Proc. Roy. Soc. (London) A400 (1985) 229

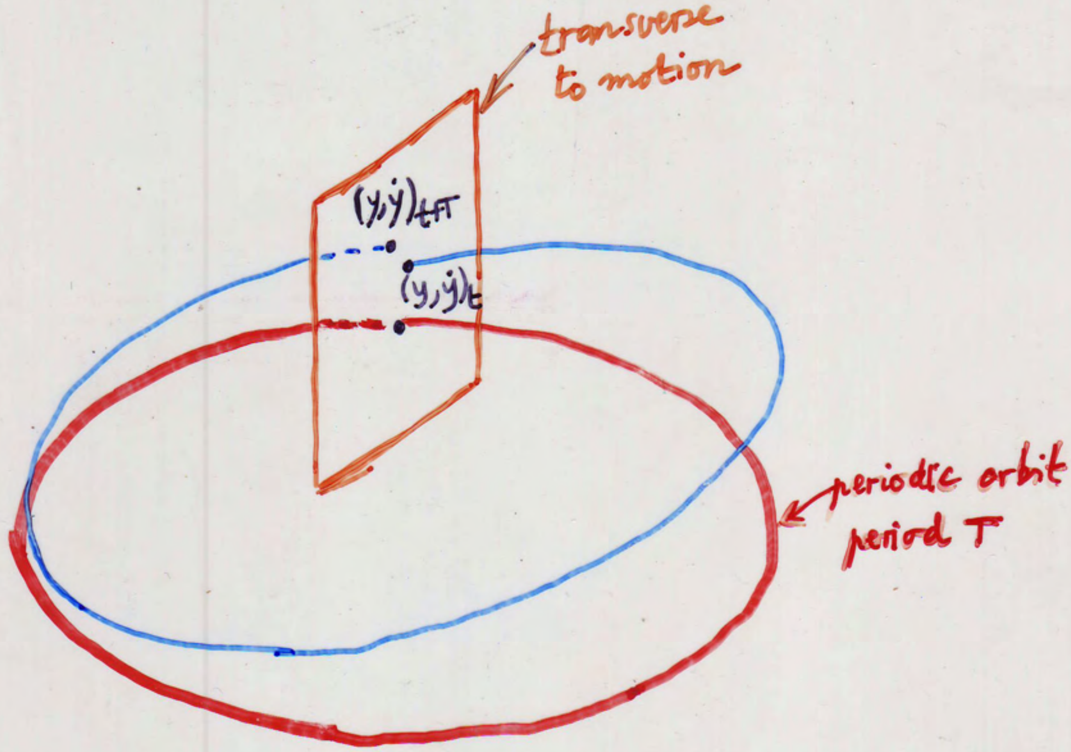
Semiclassical Theory of Spectral Rigidity

Using the density of quantum eigenvalues in terms of classical closed orbits
(Gutzwiller, Balian and Bloch)

At the finest scale of spectral structure (mean level spacing) : behaviour of extremely long orbits

For integrable systems, the orbits form continuous families whose number grows with period T as T^N

For chaotic systems, the orbits are isolated and unstable and their number proliferate exponentially $\frac{\exp^{HT}}{HT}$ H: Kolmogorov entropy ; instability exponent of the orbit



$$\begin{pmatrix} y \\ \dot{y} \end{pmatrix}_{t+T} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} y \\ \dot{y} \end{pmatrix}_t$$

M : monodromy matrix . $\det M = 1$; eigenvalues λ_1, λ_2

trajectory
- stable: $\lambda_1 = e^{i\vartheta}, \ \lambda_2 = e^{-i\vartheta} \quad (\vartheta \neq 0, \pi)$
- neutral: $\lambda_1 = \lambda_2 = 1$ or $\lambda_1 = \lambda_2 = -1$
- unstable: $\lambda_1 = \pm e^{u}, \ \lambda_2 = \pm e^{-u}$

SEMICLASSICS

$$K_{GOE}(\tau) = 2\tau - \tau \log(1+2\tau) \qquad 0 \le \tau \le 1$$

$$= 2\tau - 2\tau^2 + O(\tau^3)$$

leading off-diagonal correction

M. Sieber, K. Richter, Physica Scripta T90:128, 2001

M. Sieber, J. Phys. A35 (2002) L613

free motion on a Riemann surface of constant negative curvature

also Turek, Richter, preprint

Berkolaiko, Schanz, Whitney, Phys. Rev. Lett. 88 (2002) 104101

for generic quantum graphs

B.S.W., nlin.CD/0205014

$K(\tau) = 2\tau - 2\tau^2 + 2\tau^3 + O(\tau^4)$

for a family of quantum graphs

D. Spehner, nlin.CD/0303051

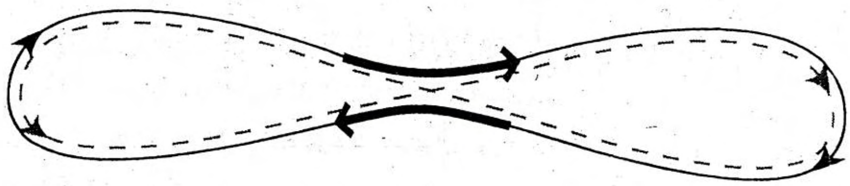
extends Sieber, Richter for general 2-dimensional hyperbolic systems

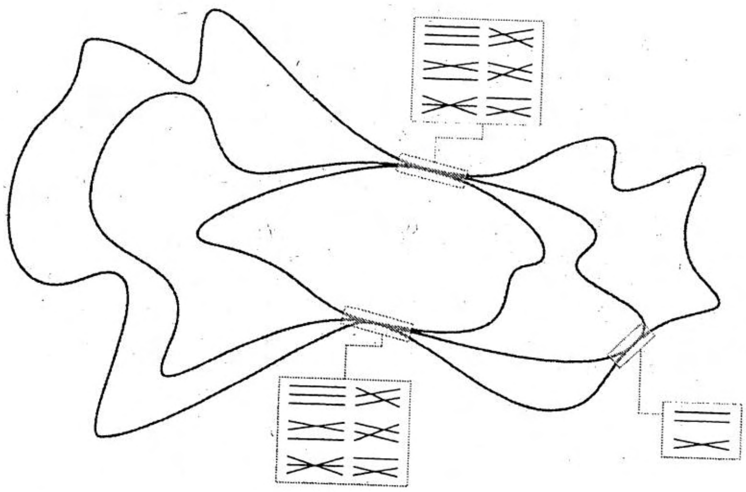
S. Müller, S. Heusler, P. Braun, F. Haake, A. Altland

'Semiclassical foundation of universality in Quantum Chaos'

nlin/CD/0401021, Phys. Rev. Lett. 93 (2004) 014103-1-4

Phys. Rev. E72 (2005) 046207





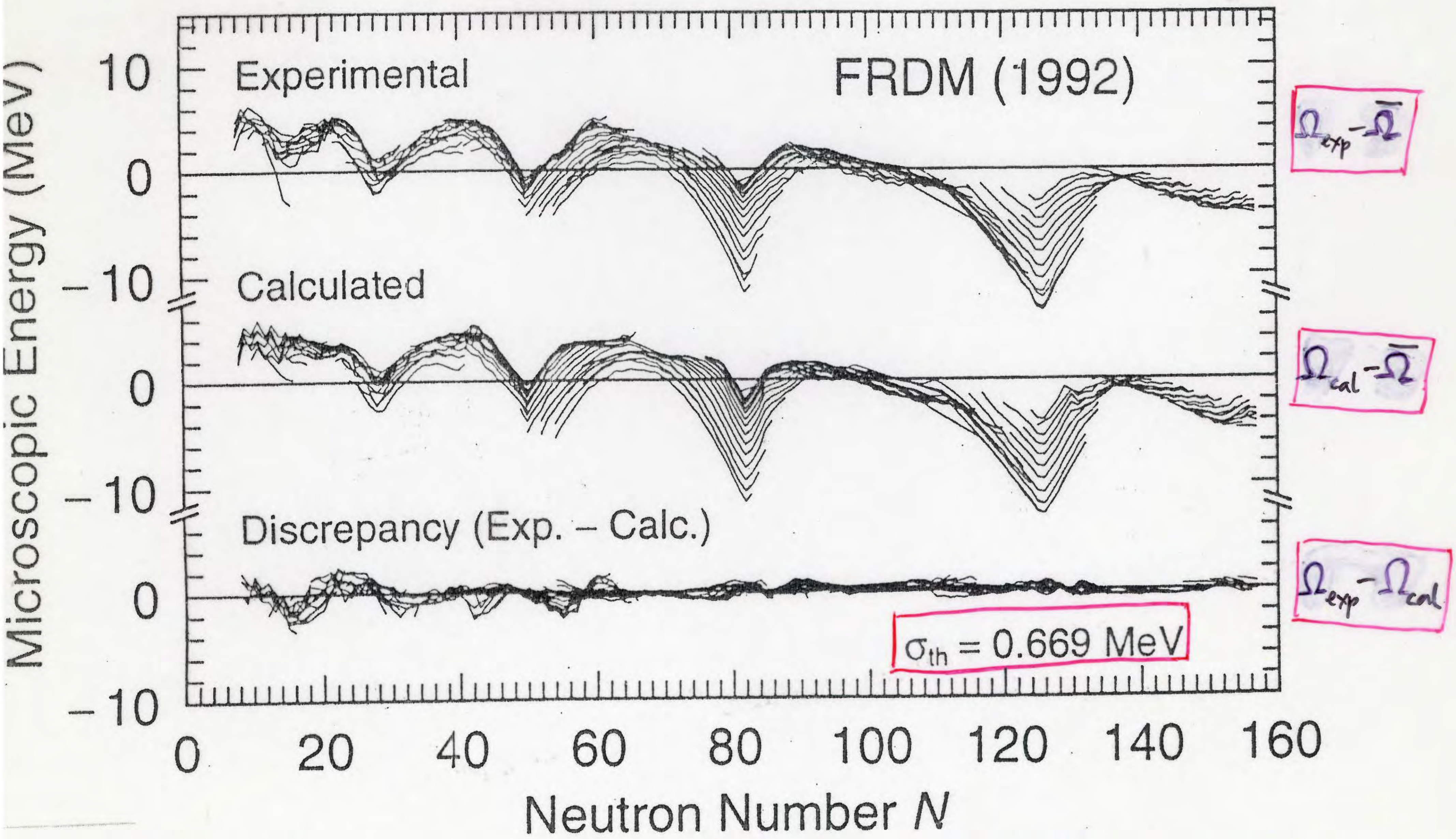


Figure 1

Möller, Nix, Myers, Swiatecki
At. Data Nucl. Data Tables 59 (1995) 185

ATOMIC NUCLEI

Level density

$$\bar{\rho}(E) \sim \frac{e^{2\sqrt{aE}}}{E^{5/4}}$$

Bethe, Non-interacting fermions

smooth

Binding energies

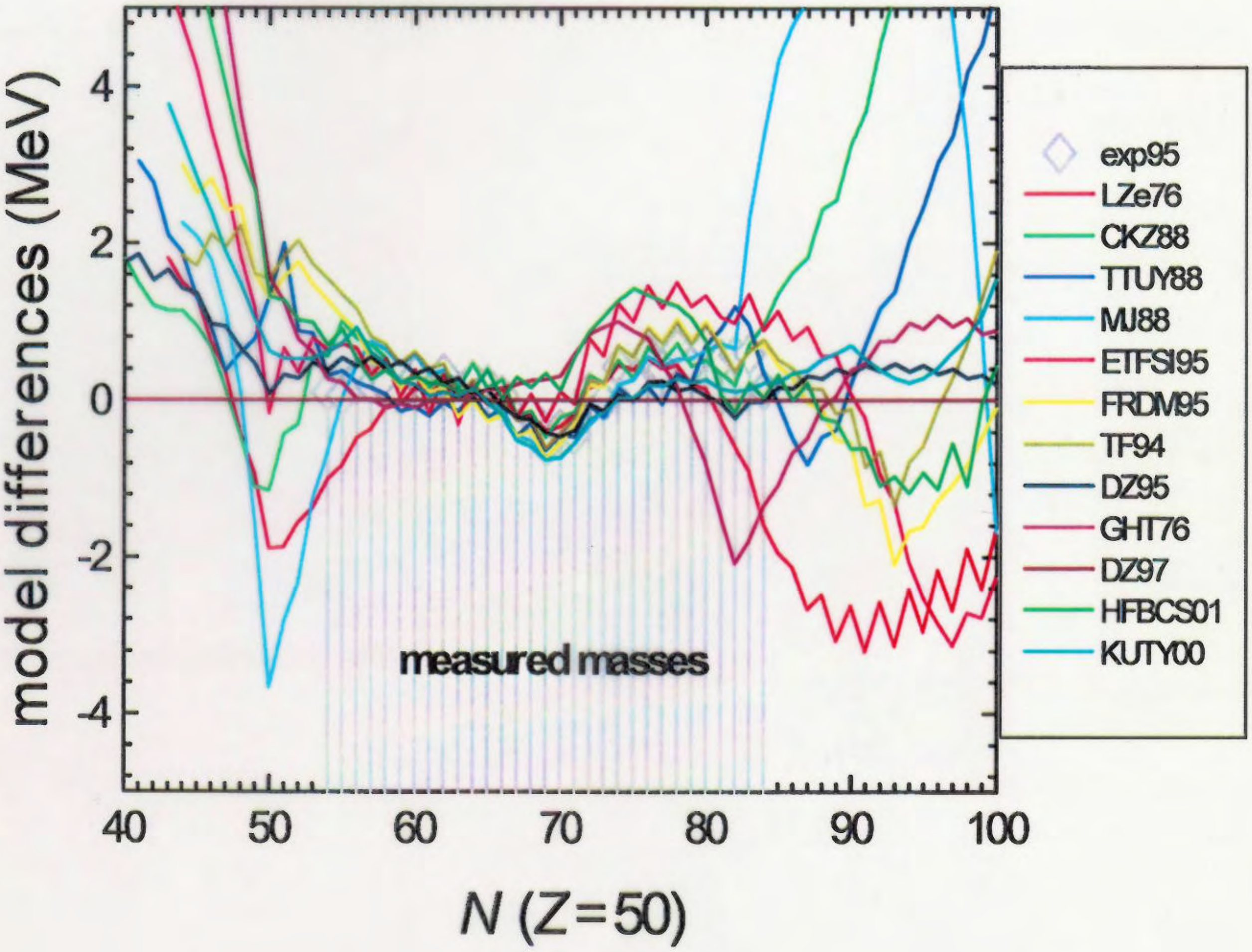
$$\bar{\Omega}(A) = b_{vol}\, A - b_{surf}\, A^{2/3} - \frac{1}{2} b_{sym} \frac{(N-Z)^2}{A} - \frac{3}{5} \frac{Z^2 e^2}{R_c}$$

volume surface symmetry Coulomb

$$A = N + Z$$

neutrons, # protons

Liquid drop, Bethe Weizsäcker



Audi, Lunney

CSNSM Orsay

$$B = \bar{B} + \tilde{B}$$

↑ liquid drop

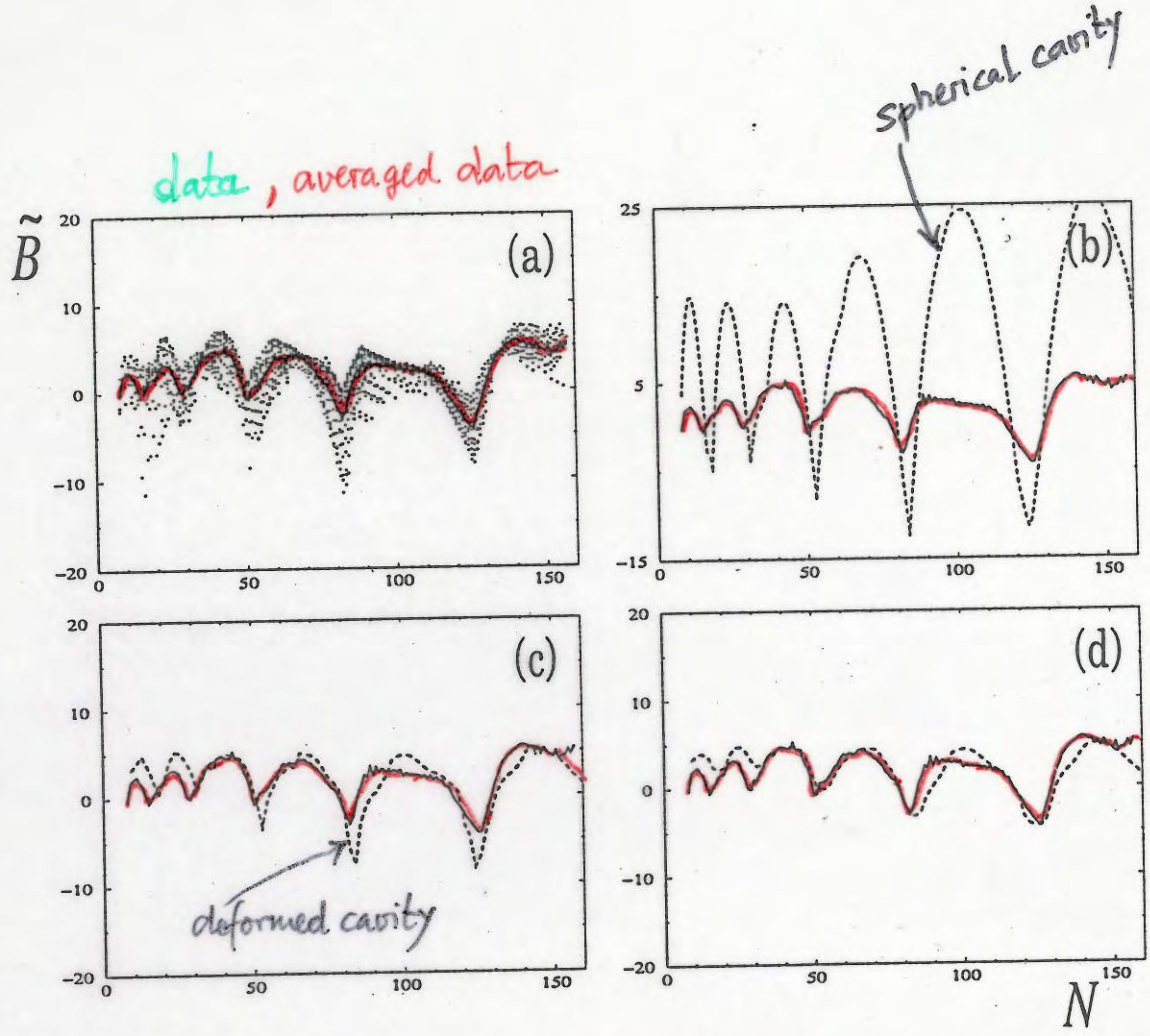


Fig. 8. Comparison between different theoretical approximations and experimental values of the fluctuating part of the nuclear masses. (a) Experimental data (dots), with their average behavior (full line). Average experimental data (full line) compared to the predictions of (dashed line) (b) a spherical cavity, (c) a (possibly) deformed cavity, and (d) a (possibly) deformed cavity with a finite coherence length.

P. Leboeuf
nucl-th/0406064

$$\Omega = \bar{\Omega} + \tilde{\Omega} = \int dE \, E \, \rho(E)$$

Periodic Orbit Theory

Gutzwiller

Balian & Bloch

General strategy

$$O = \bar{O} + \tilde{O}$$

($\bar{O}$: smooth; $\tilde{O}$: oscillating, fluctuating)

level density $\rho(E) = \sum_i \delta(E - E_i)$

oscillating part $\tilde{\rho}(E, x)$

$$\tilde{\rho}(E,x) = 2 \sum_{p} \sum_{r=1}^{\infty} A_{p,r}(E,x) \cos\left[r S_p(E,x)/\hbar + \nu_{p,r}\right]$$

periodic orbits

S_p: classical action along the primitive periodic orbit p

T_p : $\frac{\partial S_p}{\partial E}$ period of the periodic orbit

Binding energy

$$\Omega = \bar{\Omega} + \tilde{\Omega}$$

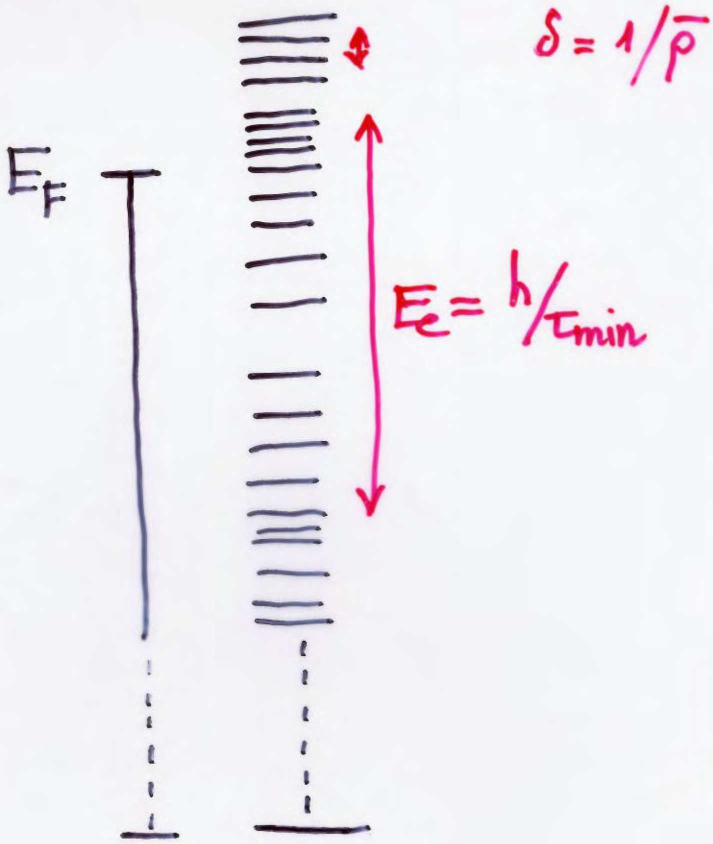
$$= \bar{\Omega} + \tilde{\Omega}_{reg} + \tilde{\Omega}_{ch}$$

↑ regular ↑ chaotic

Plausible $\langle \tilde{\Omega}_{reg} \tilde{\Omega}_{ch} \rangle = 0$

Semi classical theory for single-particle motion

$$\rho(E) = \sum_i \delta(E - E_i) = \bar{\rho} + \tilde{\rho}$$



Characteristic scales

Energy : δ, E_c

Time : τ_H, τ_{min} $\qquad \tau_H = h\bar{\rho} = h/\delta$

$$g = \frac{E_c}{\delta} \gg 1$$

is the number of fermions contained in the shell, i.e. in a window of size E_c below the Fermi energy

Variances in terms of the form factor $K(\tau)$

$$\langle \tilde{N}^2 \rangle = \frac{1}{2\pi^2} \int_0^\infty K(\tau) \, \frac{d\tau}{\tau^2}$$

$$\langle \tilde{\Omega}^2 \rangle = \frac{\hbar^2}{2\pi^2} \int_0^\infty K(\tau) \, \frac{d\tau}{\tau^4}$$

$$\tilde{\Omega}(A) = 2\hbar^2 \sum_p \sum_{r=1}^{\infty} \frac{A_{p,r}}{r^2 \tau_p^2} \cos\left(r S_p/\hbar + \nu_{p,r}\right)$$

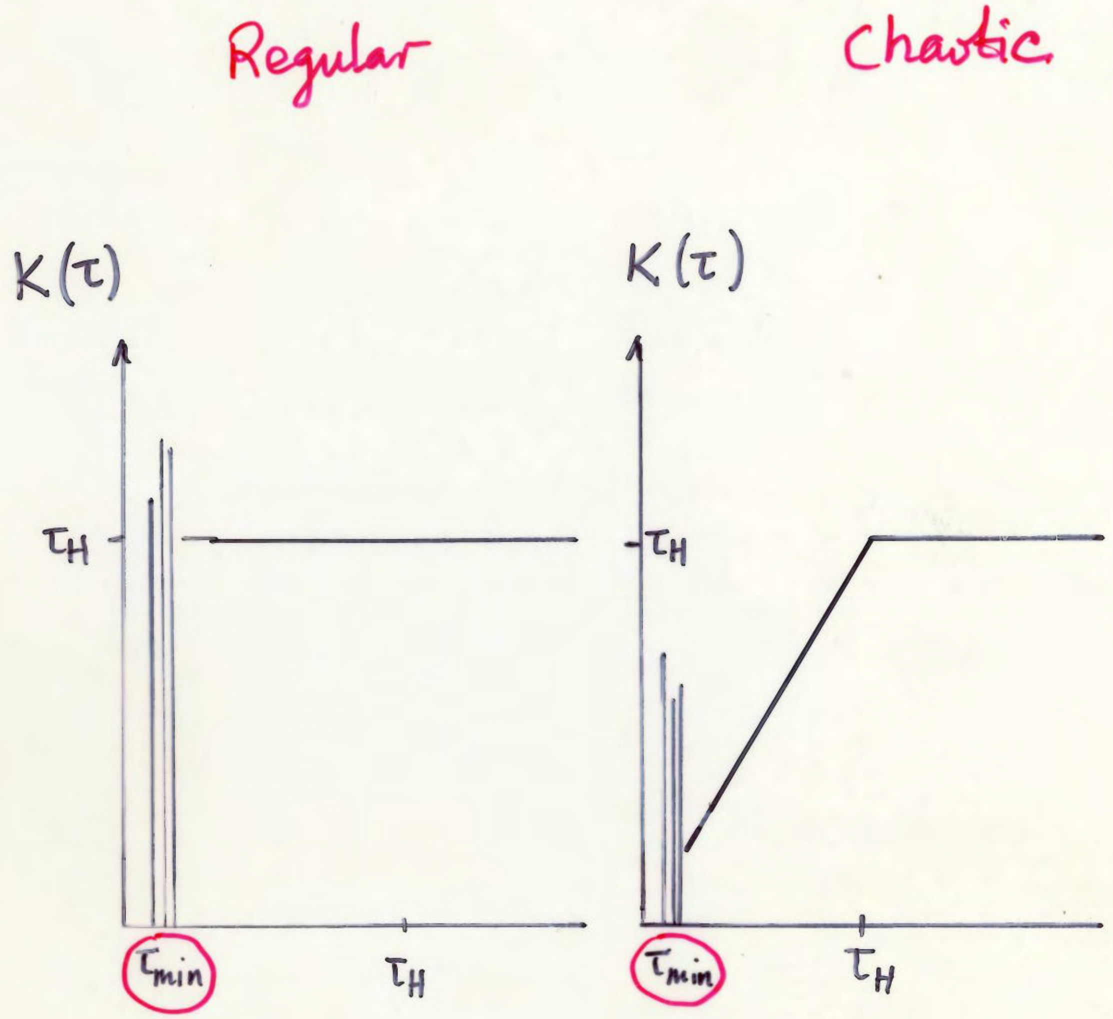
$$\langle \tilde{\Omega}^2 \rangle \approx 2\hbar^4 \sum_{p,r} \frac{A_{p,r}^2}{r^4 \tau_p^4}$$

$$K_D = \hbar^2 \sum_{p,r} A_{p,r}^2 \, \delta(\tau - r\tau_p)$$

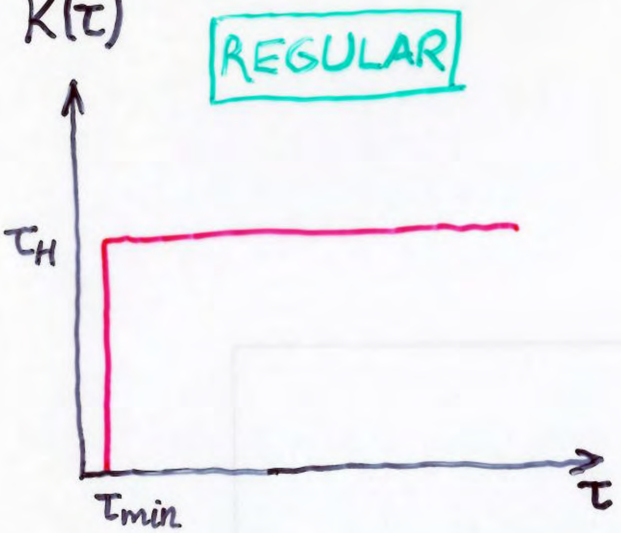
$$\langle \tilde{\Omega}^2 \rangle = \frac{\hbar^2}{2\pi^2} \int_0^{\infty} \frac{d\tau}{\tau^4} K(\tau)$$

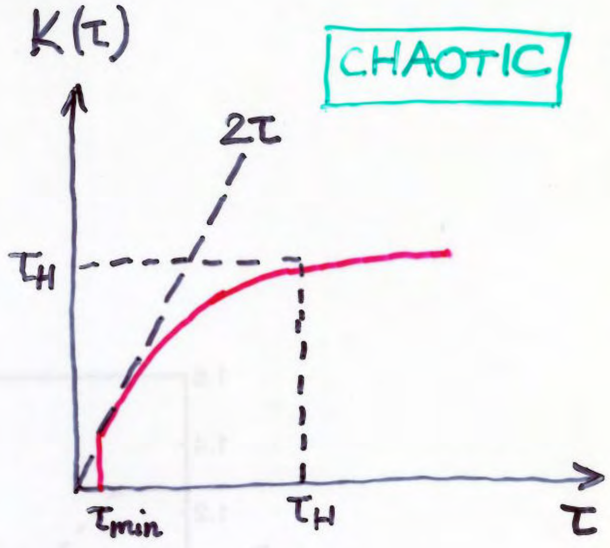
$$K_D(\tau) \approx \begin{cases} 0 & \tau < \tau_{min} \\ \tau & \tau > \tau_{min} \end{cases}$$

Schematic view of the form factor



$$\langle \tilde{\Omega}^2 \rangle = \int_0^\infty d\tau \, \frac{K(\tau)}{\tau^4}$$





$$E_c = \frac{h}{\tau_{min}} = \frac{77.5}{A^{1/3}} \text{ MeV}$$

$$\delta \approx \frac{2 E_F}{3A} = \frac{25}{A} \text{ MeV}$$

$$g = \frac{E_c}{\delta} = \pi A^{2/3}$$

$$\langle \tilde{\Omega}^2_{reg} \rangle = \frac{1}{24\pi^4} \, g \, E_c^2 \qquad \langle \tilde{\Omega}^2_{ch} \rangle = \frac{1}{8\pi^4} E_c^2$$

$$\sigma_{reg} = \sqrt{\langle \tilde{\Omega}^2_{reg} \rangle} = 2.8 \text{ MeV} \qquad \sigma_{ch} = \sqrt{\langle \tilde{\Omega}^2_{ch} \rangle} = \frac{2.78}{A^{1/3}} \text{ MeV}$$

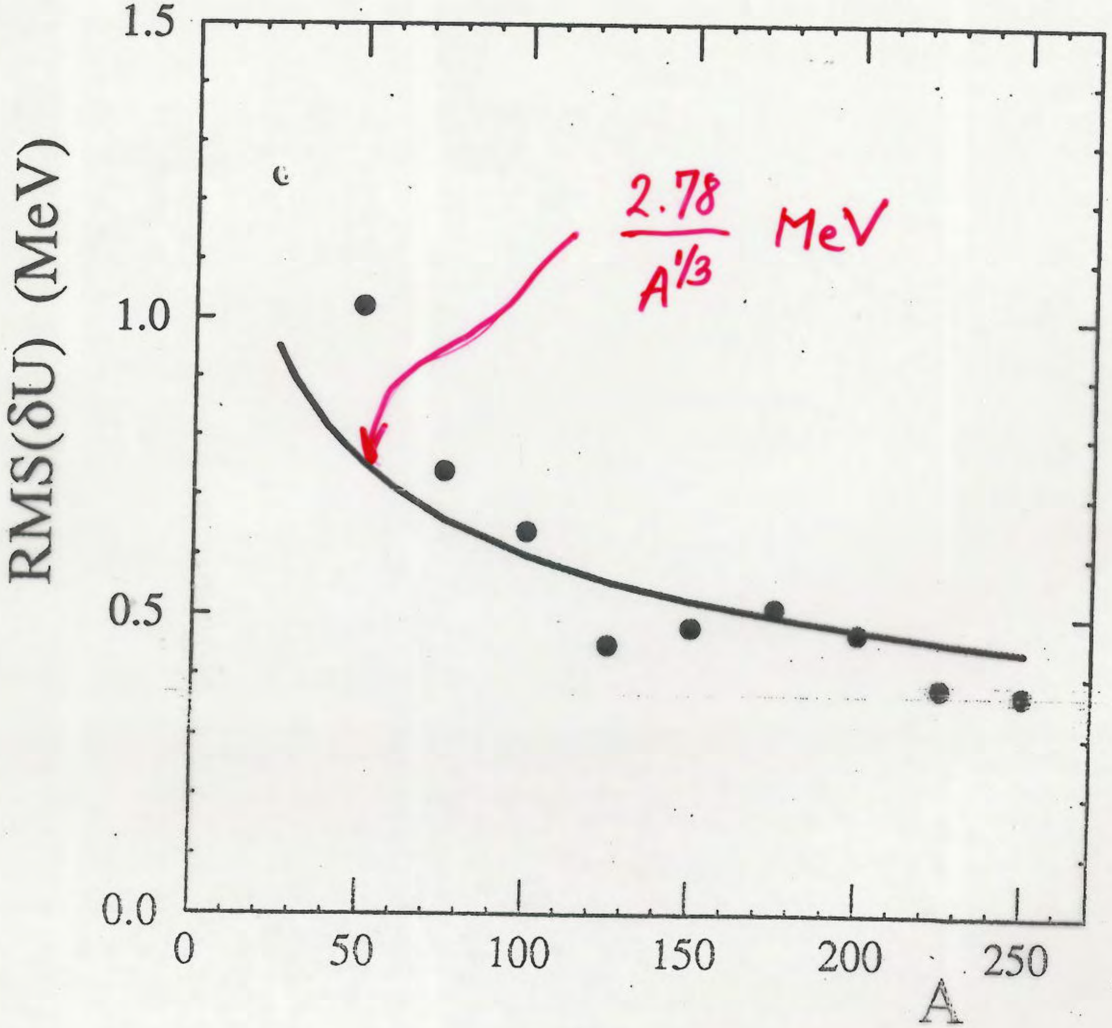


FIG. 1. RMS of the difference δU between computed and observed masses as a function of mass number A. Dots taken from Fig. 7 of Ref. [3], solid curve from Eq. (15).

Bohigas & Leboeuf
PRL 88 (2002) 092502

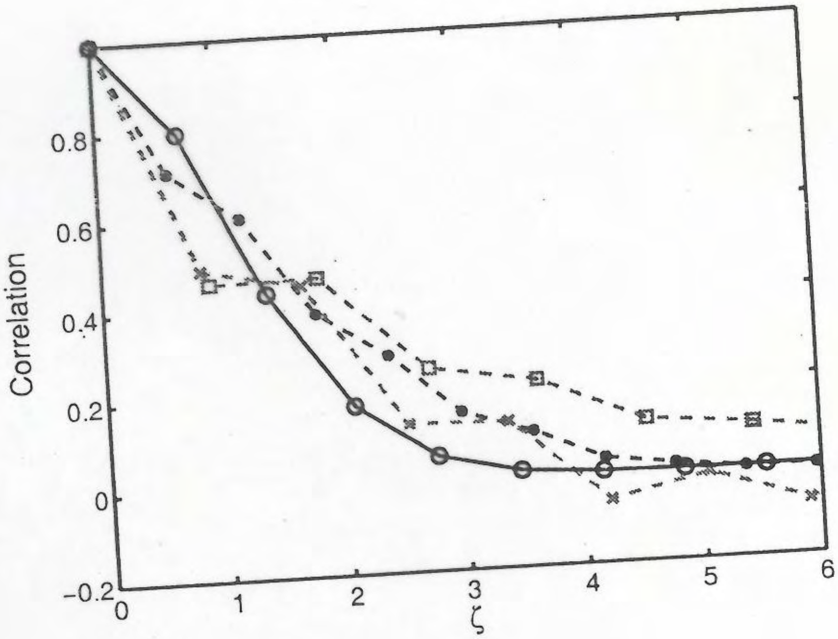
correlation function

$$C_Z(dN) = \frac{\langle \tilde{\Omega}(Z,N)\,\tilde{\Omega}(Z,N+dN)\rangle_N}{\langle \tilde{\Omega}^2 \rangle_N}$$

$\tilde{\Omega}(Z,N)$: difference between calculated and measured masses for nucleus having Z protons, N neutrons

dN : difference in neutron number along isotopic chain

For a specific dN and Z every $\tilde{\Omega}(Z,N)\,\tilde{\Omega}(Z,N+dN$ is calculated



. 2: Correlation of the mass errors between neighbouring
ɪpes as a function of the dimensionless parameter ζ. Cir-
squares and crosses correspond to the same models as in
1. The solid line is the theoretical result Eq.(12).

Olofsson, Åberg, Bohigas, Leboeuf, ~~submitted~~ Phys. Rev. Lett. 96 (2006) 042502

Look, however, at

- J. Barea, A. Frank, J. G. Hirsch, P. Van Isacker
  'Nuclear masses set bounds on quantum chaos'
  nucl-th/0502038

- A. Molinari, H. A. Weidenmüller
  'Statistical fluctuations of nuclear ground-state energies and binding energies'
  nucl-th/0403028
  Phys. Lett. B601 (2004) 119

  'Nuclear Masses, Chaos and the Residual Interaction'
  nucl-th/06
  Phys. Lett. B (2006)

Nuclear Physics News 17 # 2, 2007, p. 31

George F. Bertsch

U.S. D.o.E.

SciDAC (Scientific Discovery through Advanced Computing)

Project: 'Building a Universal Nuclear Energy Density Functional' UNEDF

UNEDF is founded at $ 3M / year

Five year duration

Rydberg atom: Highly excited state of an atom

Exotic properties:

Gigantic: diameter $\sim \frac{1}{100}$ mm $\sim 10^4$ Bohr radius

Long lived: $10^{-3} - 1$ sec to be compared to 10^{-7} s

Response to EB fields

Rydberg atoms $\approx$ hydrogen atom

$$E_n = -\frac{k}{n^2} \qquad k \simeq 13.6\,eV$$

~~earth~~ outer space $n \lesssim 350$

$n \lesssim 100$

$$R \sim n^2$$

$$D_n \sim \frac{1}{n^3}$$

$$\ell = 0, 1, \ldots, n-1$$

$$\rho \sim n^5$$

Experimentally: Rydberg atoms can be created by

- Bombardment of gas with charged particles
- Tunable laser to excite atoms

Diamagnetic Kepler problem

$$H = \frac{p^2}{2} - \frac{1}{r} + \frac{\gamma}{2} L_z + \frac{\gamma^2}{8}(x^2+y^2)$$

$$r = x^2+y^2+z^2$$

$\gamma = B/B_c$ $B_c = 2.35 \times 10^5\ T$

vector potential $\vec{A} = -1/2\,(\vec{r} \times \vec{B})$ (symmetic Gauge)

Energy and L_z constants of the motion

Limiting cases

$\gamma \to 0$ ← Coulomb

$\gamma \to \infty$ → Landau

Interesting regime

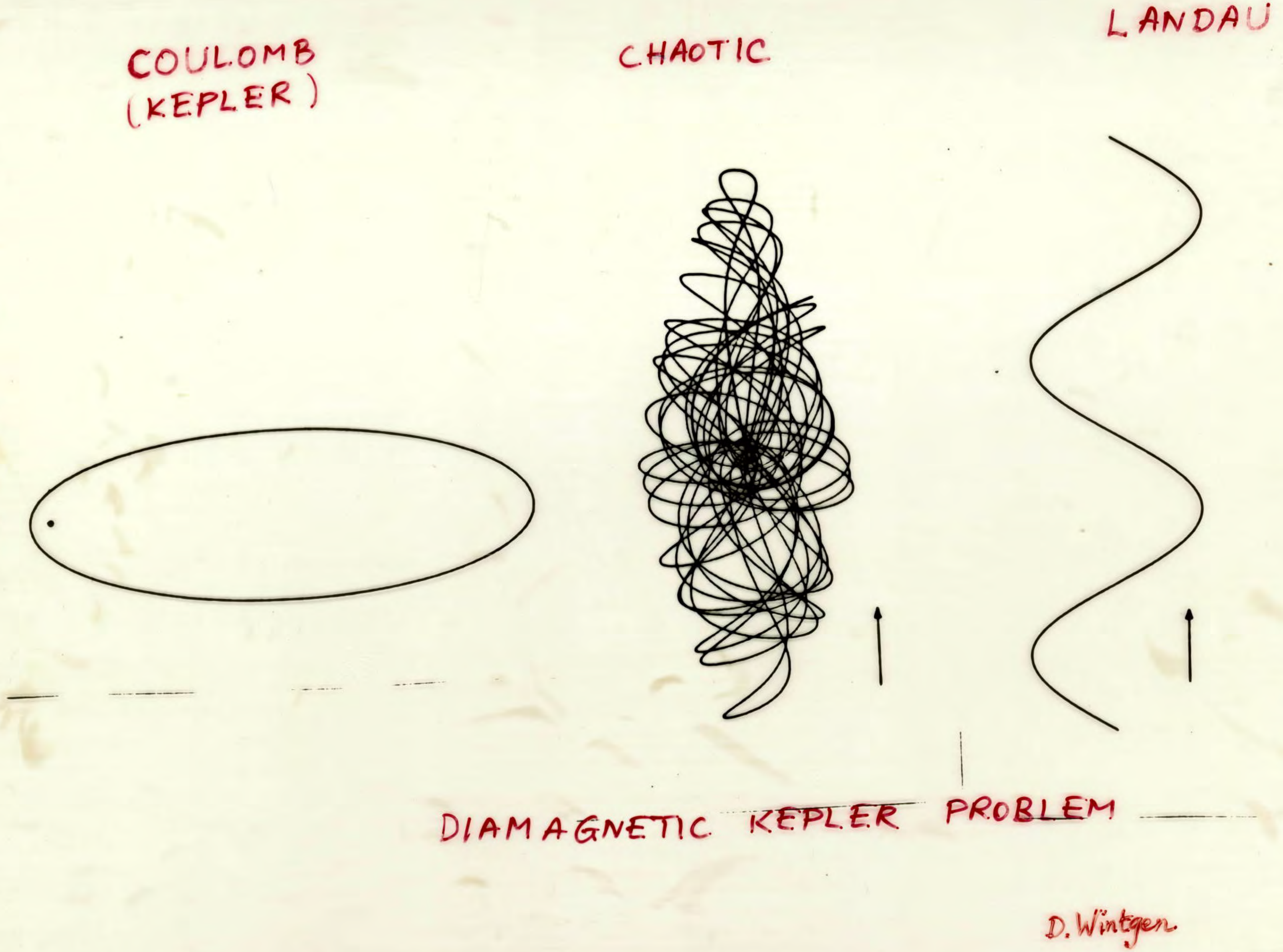
Coulomb force $\simeq$ Lorentz force

Parameter of the problem

$$\beta = \frac{\gamma^2}{2E^3}$$

Classically

	Regular	Transition	Fully Chaotic
β	$\lesssim 1$		$\gtrsim 60$


COULOMB
(KEPLER)
CHAOTIC
LANDAU
DIAMAGNETIC KEPLER PROBLEM
D. Wintgen

 D. Delande

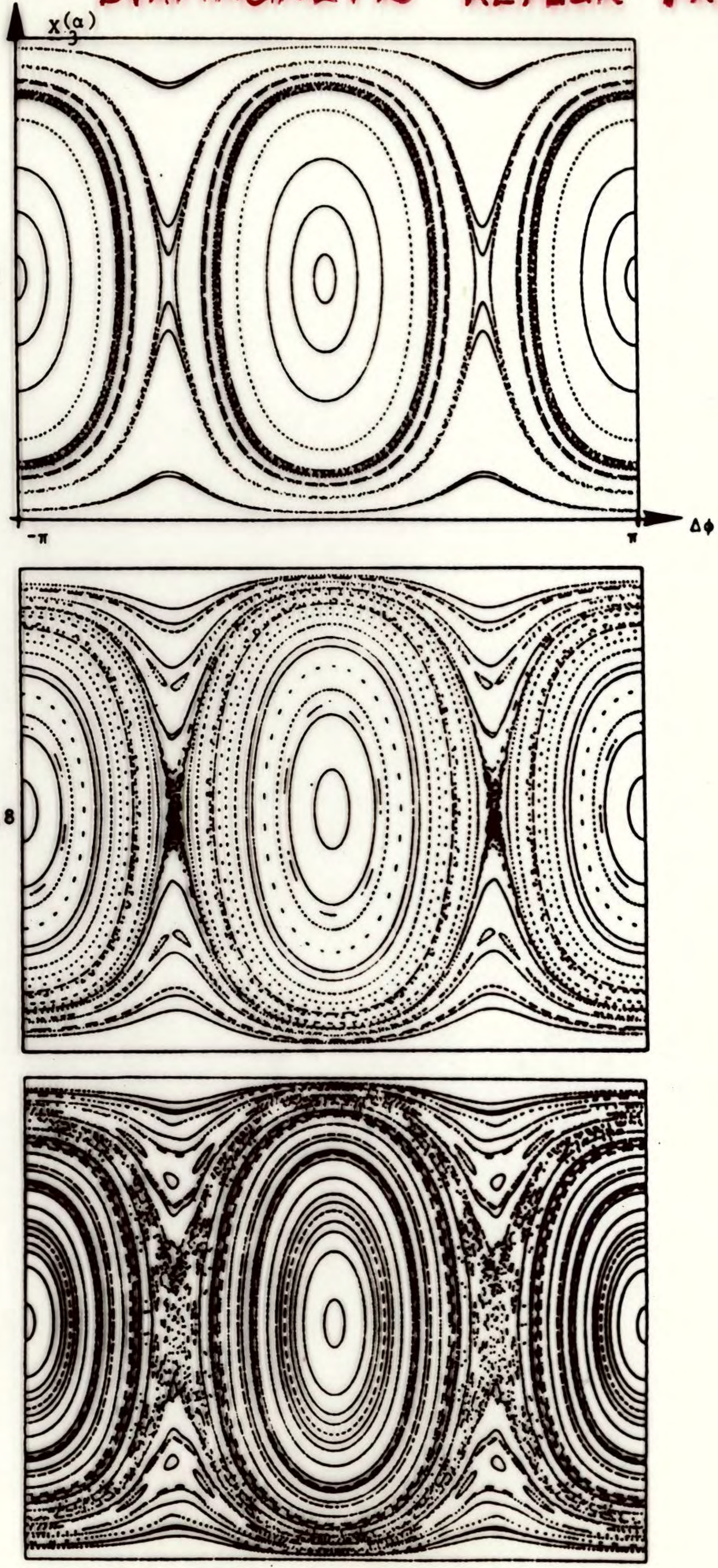
DIAMAGNETIC KEPLER PROBLEM

$\alpha = \frac{1}{\sqrt{-2E}}$

$\beta = 0.2$ — β = 0,2

$\beta = 0.8$ — β = 0,8

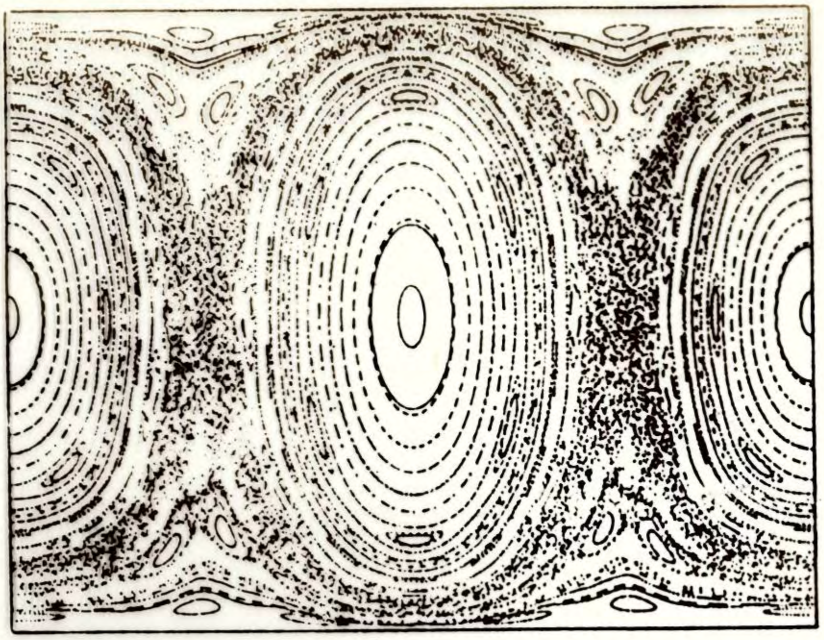
$\beta = 1$ — β = 1



D.Delande

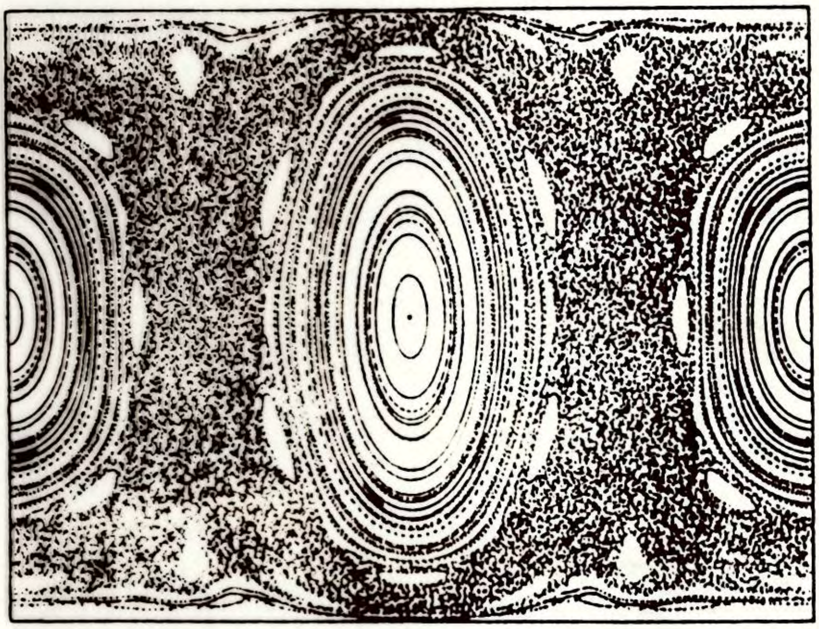
β=1.35

β = 1,35



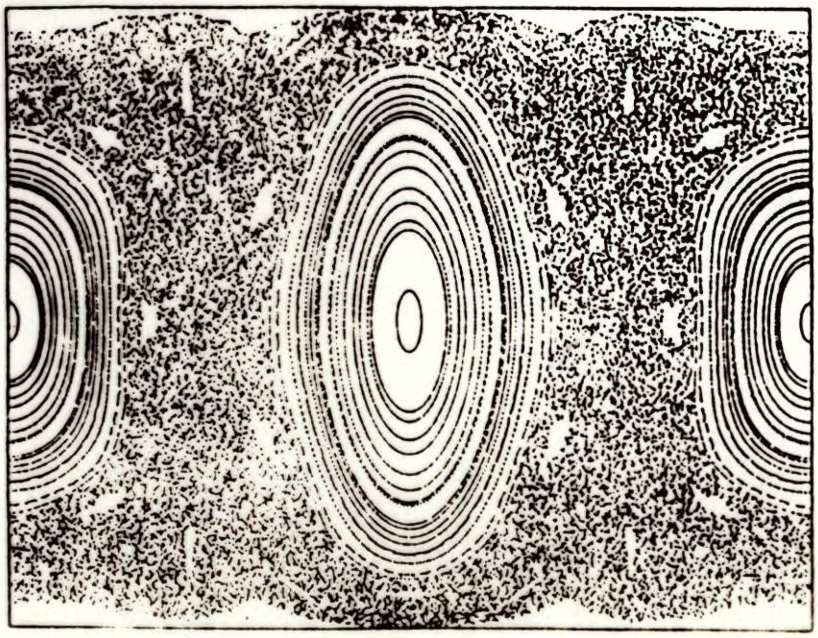
β=2

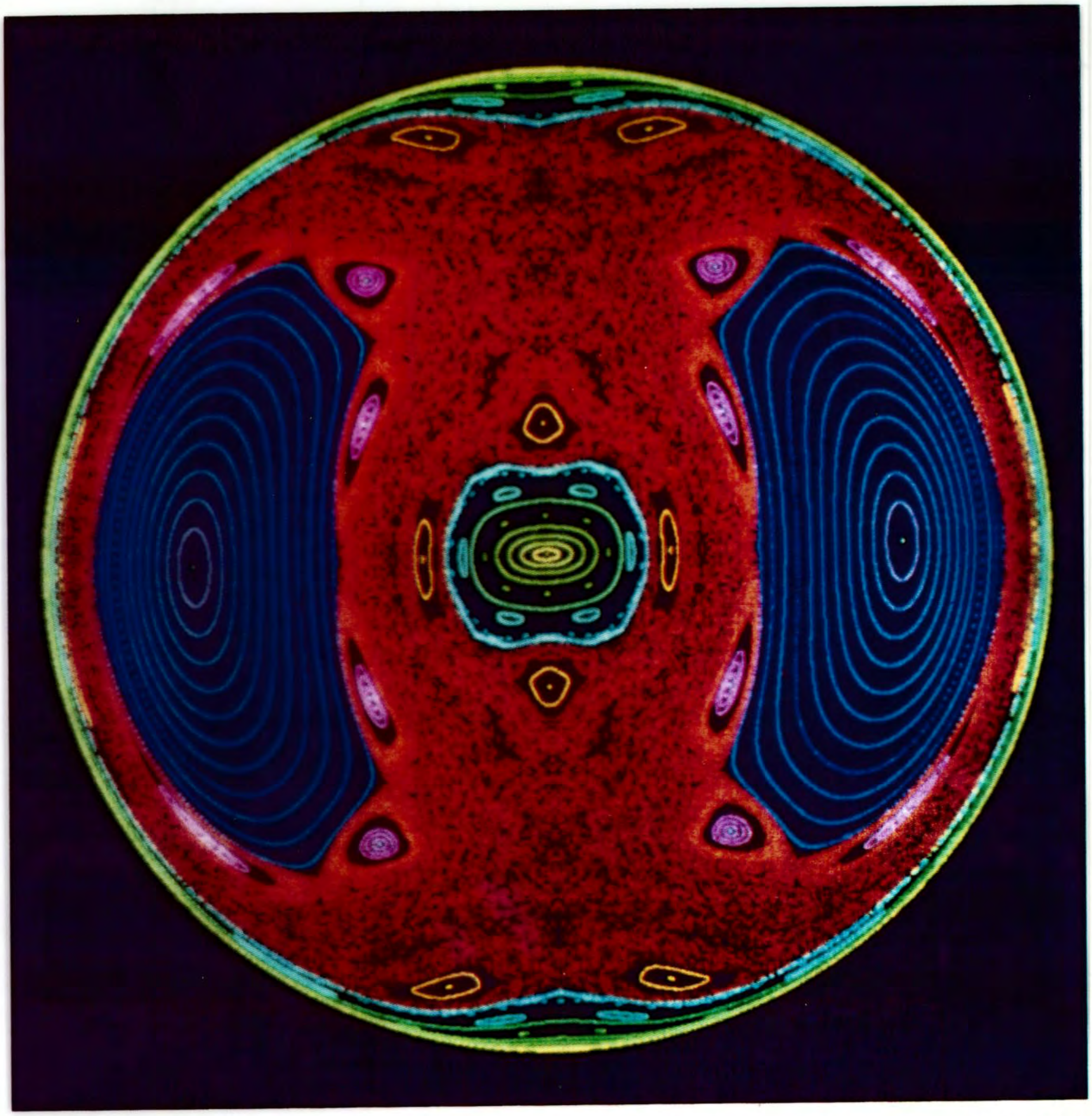
β = 2



β=2.5

β = 2,5





POINCARÉ SECTION OF A HYDROGEN ATOM in a strong magnetic field has regions (*orange*) where the points of the electron's trajectory scatter wildly, indicating chaotic behavior. The section is a slice out of phase space, an abstract six-dimensional space: the usual three for the position of a particle and an additional three for the particle's momentum.

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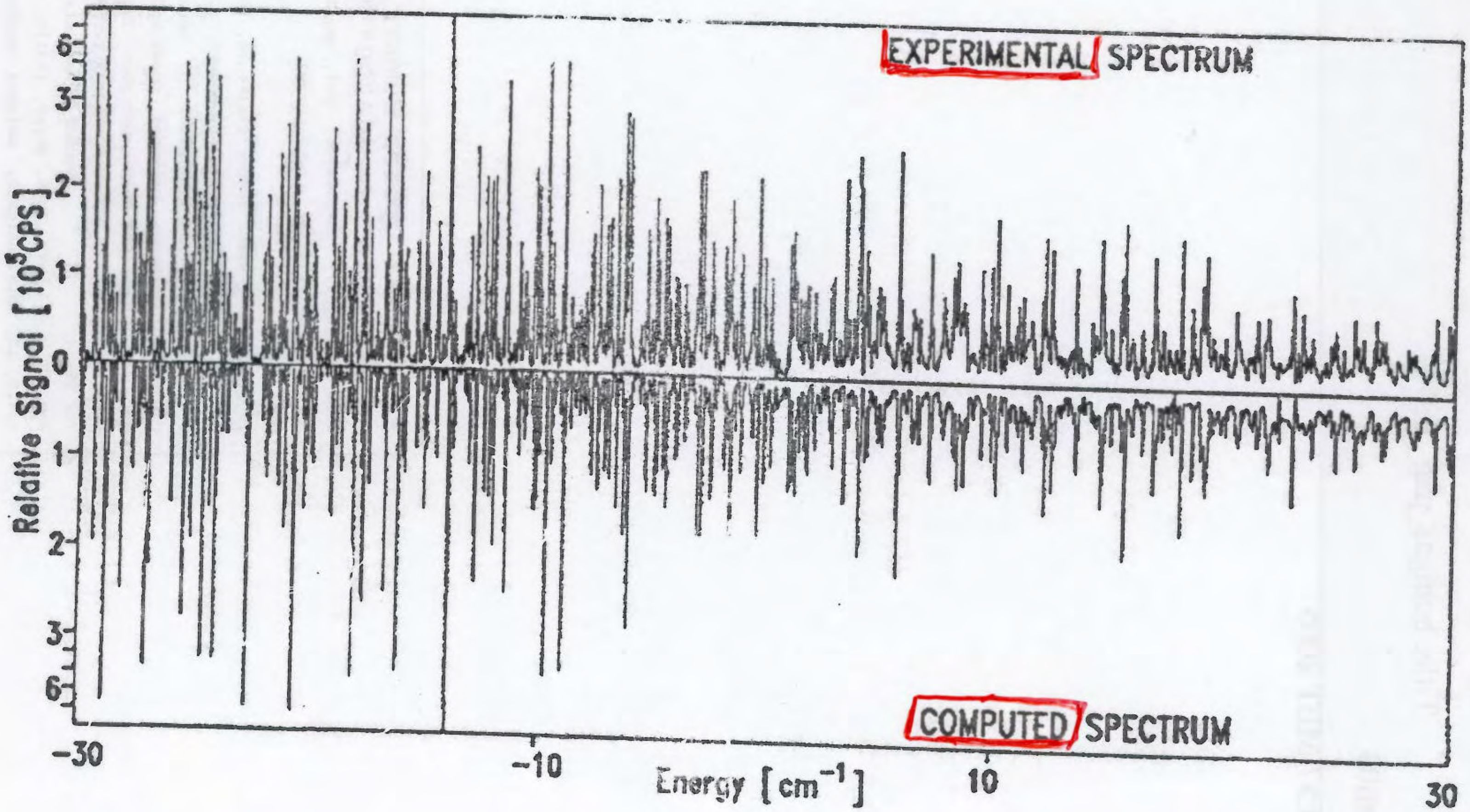

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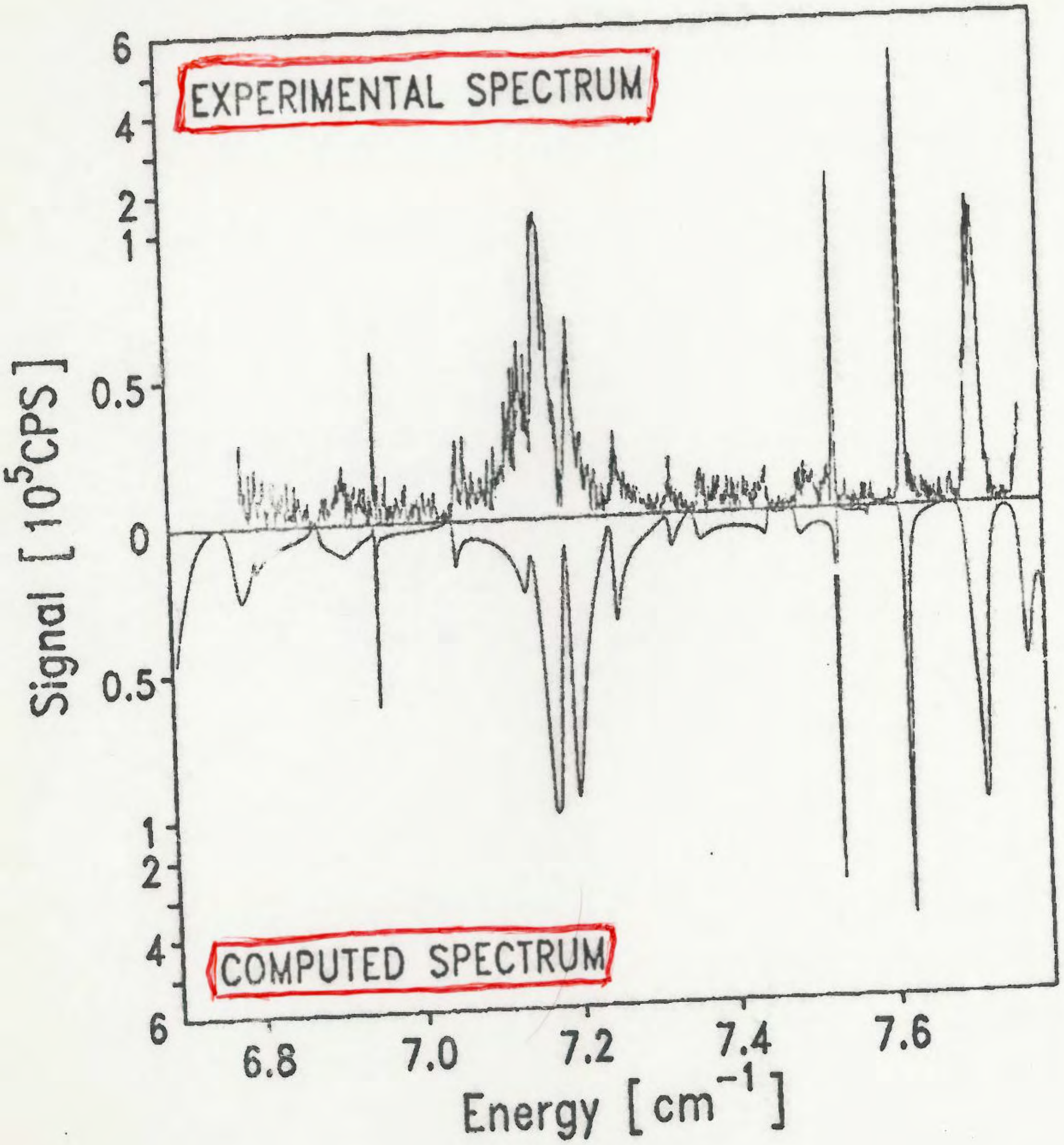

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