Homework 4 Answers

1. a) Geodesics:

$$\left(1-\frac{2m}{r}\right)\left(\frac{dt}{d\lambda}\right) = E \qquad -(1)$$

$$r^2\frac{d\phi}{d\lambda} = L \qquad -(2)$$

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}E^2 \qquad -(3)$$

$$V(r) = \frac{1}{2}\varepsilon - \frac{\varepsilon m}{r} + \frac{L^2}{2r^2} - \frac{mL^2}{r^3}$$

$\varepsilon = 1$ for particles.

Circular orbits $r =$ const

$$\Rightarrow \frac{dr}{d\lambda} = 0 \qquad (*)$$

and

$$\frac{d^2r}{d\lambda^2} = 0 \qquad (**)$$

$(**)$ is essential --- $(*)$ alone will only locate the perhelion / aphelion.

Take $\frac{d}{dr}$ of (3)

$$\underbrace{\frac{dr}{d\lambda}\frac{d}{dr}\left(\frac{dr}{d\lambda}\right)}_{=\frac{d^2r}{d\lambda^2}\equiv 0} + \frac{dV}{dr} = 0$$

Circular orbit: $\frac{dV}{dr} = 0$, $V(r) = \frac{1}{2}E^2$

$$\frac{dV}{dr} = 0 = \frac{m}{r^2} - \frac{L^2}{r^3} + \frac{3mL^2}{r^4} = 0$$

$$\Leftrightarrow \frac{m}{r^4}\left(r^2 - \frac{L^2}{m}r + 3L^2\right) = 0$$

Circular orbit:

$$r = \frac{L^2}{2m} \pm \frac{1}{2m}\left(L^4 - 12m^2L^2\right)^{1/2}$$

or
$$L^2 = \frac{mr^2}{(r-3m)} \qquad -(4)$$

which is the first of the three desired equations

However, on a circular orbit we also have:

$$V(r) = \tfrac{1}{2}E^2$$

$$\Rightarrow \quad E^2 = 2V(r) = 1 - \frac{2m}{r} + \frac{L^2}{r^2} - \frac{2mL^2}{r^3}$$

$$= 1 - \frac{2m}{r} + \frac{m}{(r-3m)} - \frac{2m^2}{r(r-3m)}$$

$$= \frac{1}{r(r-3m)}\left[r(r-3m) - 2m(r-3m) + mr - 2m^2\right]$$

$$= \frac{(r-2m)^2}{r(r-3m)} \qquad -(5)$$

$$r^2\frac{d\phi}{dt} = r^2\frac{d\phi}{d\lambda} \Big/ \frac{dt}{d\lambda} = \frac{L\left(1-\frac{2m}{r}\right)}{E}$$

$$\Rightarrow \quad \omega^2 = \left(\frac{d\phi}{dt}\right)^2 = \frac{L^2}{E^2}\,\frac{1}{r^4}\left(1-\frac{2m}{r}\right)^2$$

$$= \frac{mr^2}{\cancel{(r-3m)}} \cdot \frac{\cancel{(r-3m)}\,r}{\cancel{(r-2m)^2}} \cdot \frac{1}{r^6}\,\cancel{(r-2m)^2}$$

$$= \frac{m}{r^3}$$

– See p. 7. for rest of 1 a).

b) Circular orbits have

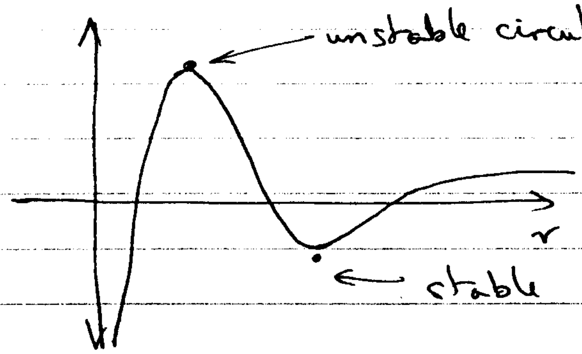
$$r = \frac{L^2}{2m}\left[1 \pm \left(1 - \frac{12m^2}{L^2}\right)^{1/2} \right]$$

There are two roots $\Leftrightarrow$ two circular orbits if $L^2 > 12m^2$

There is one root $\Leftrightarrow$ one circular orbit if $L^2 = 12m^2$

There are no roots $\Leftrightarrow$ no circular orbits if $L^2 < 12m^2$.

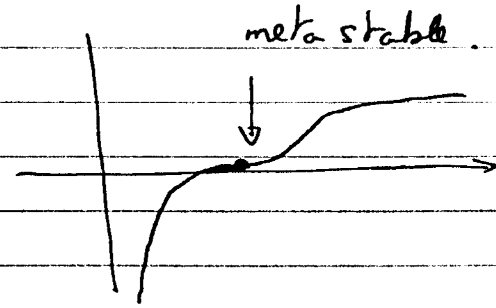
If L^2/m^2 one has



$$r = r_+ = \frac{L^2}{2m}\left[1 + \left(1 - \frac{12m^2}{L^2}\right)^{1/2}\right] \quad \text{is stable}$$

$$r = r_- = \frac{L^2}{2m}\left[1 - \left(1 - \frac{12m^2}{L^2}\right)^{1/2}\right] \quad \text{is unstable.}$$

If $L^2 = 12m^2$, then the one orbit at $r = L^2/2m$ is metastable.
- --- or marginally unstable



c) $$r_+ = \frac{L^2}{2m} + \frac{1}{2m}\left(L^4 - 12m^2L^2\right)^{1/2}$$

$$= \frac{m}{2}\left[x + (x^2 - 12x)^{1/2}\right] \qquad \text{where } x = \frac{L^2}{m^2}$$

is the radius of its stable, or meta stable orbit.

Moreover, $$\frac{dr_+}{dx} = \frac{m}{2}\left[1 + \frac{(x-6)}{(x^2-12x)^{1/2}}\right]$$

$$= \frac{m}{2}\left[1 + \frac{(x-6)}{\left[(x-6)^2 - 36\right]^{1/2}}\right]$$

$$> 0 \qquad \text{for } x > 12$$

Thus r_+ is minimal for $x = 12 \Rightarrow L^2 = 12m^2$

$$\Rightarrow \quad r_+ = 6m$$

$$\Rightarrow \quad E^2 = \frac{(r-2m)^2}{r(r-3m)} = \frac{8}{9}$$

$$\Rightarrow \quad E = \frac{2\sqrt{2}}{3}$$

d) $$\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}E^2$$

$$V(r) = \frac{1}{2} - \frac{m}{r} + \frac{L^2}{2r^2} - \frac{mL^2}{r^3} \to \frac{1}{2} \quad \text{as } r \to \infty$$

We want $\left(\frac{dr}{d\lambda}\right) \to 0$ as $r \to \infty$ for "just escaping"

$$\Rightarrow \quad E = 1$$

- For a particle at rest at $r = 2m$:

$$\frac{d\phi}{d\lambda} = 0 \quad \Rightarrow \quad L = 0$$

$$\frac{dr}{d\lambda} = 0 \quad \Rightarrow \quad V(r) = \tfrac{1}{2} E^2$$

Thus $\quad \frac{1}{2} - \frac{m}{r} = \frac{1}{2} E^2$

$$\Rightarrow \quad \tfrac{1}{2}\left(1 - \frac{2m}{r}\right) = \tfrac{1}{2} E^2$$

$$\Rightarrow \quad E = 0$$

- Alternatively, $\left(1 - \frac{2m}{r}\right)\frac{dt}{d\lambda} = E \Rightarrow E = 0$ at $r = 2m$ provided $\frac{dt}{d\lambda}$ is finite. However, the coordinate t is singular at $r = 2m$, and so this argument is suspect. There are coordinate systems in which r & ϕ are regular at $r = 2m$ and so the argument above is better.

e) $$r_+ = \frac{m}{2}\left(x + (x^2 - 12x)^{1/2}\right) \qquad x = \frac{L^2}{m^2}$$

$$V(r) = \frac{1}{2} - \frac{m}{r} + \frac{L^2}{2r^2} - \frac{mL^2}{r^3}$$

$$\frac{dV(r)}{dr} = +\frac{m}{r^2} - \frac{L^2}{r^3} + \frac{3mL^2}{r^4} = 0$$

$$\frac{d^2V}{dr^2} = -\frac{2m}{r^3} + \frac{3L^2}{r^4} - \frac{12mL^2}{r^5}$$

$$= -\frac{m}{r^5}\left(2r^2 - \frac{3L^2}{m} r + 12L^2\right)$$

On the circular orbit:

$$L^2 = \frac{m r^2}{(r - 3m)}$$

$$\Rightarrow \quad \left.\frac{d^2V}{dr^2}\right|_{r=r_+} = -\frac{m}{r_+^5}\left[2r_+^2 - \left(\frac{3r_+}{m} - 12\right)\frac{m r_+^2}{(r_+ - 3m)}\right]$$

$$= +\frac{m}{r_+^3}\,\frac{(r_+ - 6m)}{(r_+ - 3m)} \equiv \omega_r^2$$

Note that this is positive for $r_+ > 6m$.

Thus near the minimum of $V(r)$, the radial equation is of the form:

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \left(V(r_+) - \frac{1}{2}E^2\right) + \frac{1}{2}(\omega_r^2)(r - r_+)^2 = 0$$

$$\Rightarrow \quad \left(\frac{d^2r}{d\tau}\right) + \omega_r^2\,(r - r_+) = 0$$

$$\Rightarrow \quad (r - r_+) = \alpha \cos(\omega_r(\tau - \tau_0))$$

for some constants ω_r and τ_0.

$$\text{and} \quad \omega_r \equiv \left(\frac{m}{r_+^3}\left(\frac{r_+ - 6m}{r_+ - 3m}\right)\right)^{1/2}$$

Now $$\omega_\phi = \frac{d\phi}{d\tau} = \frac{L}{r^2} = \left(\frac{m r^2}{(r - 3m)}\right)^{\frac{1}{2}}\frac{1}{r^2}$$

$$\Rightarrow \quad \omega_\phi = \left(\frac{m}{r_+^2(r_+ - 3m)}\right)^{1/2}$$

$$\omega_p = \omega_\phi - \omega_r$$

$$= \left(\frac{m}{r_+^3(r_+-3m)}\right)^{\frac{1}{2}}\left[\, 1 - \left(1-\frac{6m}{r_+}\right)^{1/2}\right]$$

is the precession frequency of the orbit.

For small m:

$$\omega_p \approx \left(\frac{m}{r_+^3}\right)^{1/2}\left(\frac{3m}{r_+}\right)$$

$$= \omega_\phi \left(\frac{3m}{r_+}\right)$$

↑ Kepler frequency of orbit.

$$\Rightarrow \frac{\omega_p}{\omega_\phi} = \frac{3m}{r_+} \rightarrow \Delta\phi\big|_{orbit} = 2\pi.\left(\frac{3Gm}{c^2 r_+}\right)$$

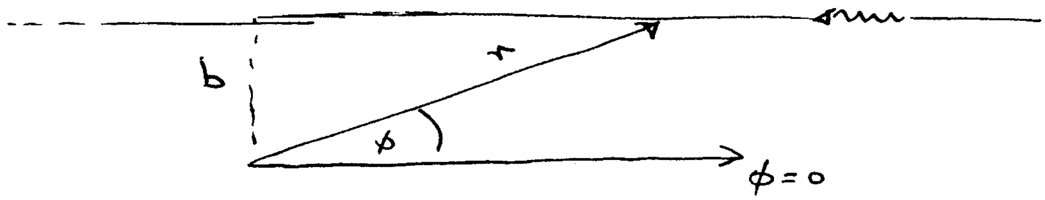
$$= \frac{6\pi Gm}{r_+ c^2}.$$

which agrees with the result in lectures

1a) continued $\underline{r = 3m}$ is a closed photon orbit

$$\left.\frac{dV}{dr}\right|_{\varepsilon=0} = -\frac{L^2}{r^2} + \frac{3mL^2}{r^3} = 0 \text{ for } r = 3m.$$

2. a)



Initial orbit : straight line with

$$\left.\begin{aligned} r\sin\phi &= b. \\ r\cos\phi &= -ct \\ &= -t \end{aligned}\right\} \text{ at } r = \infty$$

To lowest order in ϕ we have

$$\therefore \phi = b/r \quad ; \quad r = -t$$

$$\Rightarrow \phi \approx -b/t$$

$$\Rightarrow \left(\frac{d\phi}{dt}\right) \approx +b/t^2 \approx +b/r^2$$

$$\Rightarrow r^2\left(\frac{d\phi}{dt}\right) = b$$

But, for large r,

$$\left.\begin{aligned} \frac{dt}{d\lambda} &\approx E \\ \frac{d\phi}{d\lambda} &= L/r^2 \end{aligned}\right\} \Rightarrow r^2\frac{d\phi}{dt} = \frac{L}{E}$$

Thus $b = L/E$.

b) Radial motion

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{2}\left(E^2 - \frac{L^2}{r^2} + \frac{2mL^2}{r^3}\right)$$

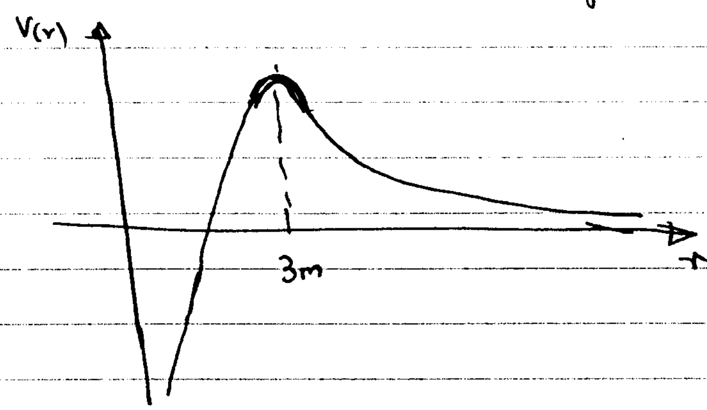
$$= \frac{1}{2}E^2\left(1 - \frac{b^2}{r^2} + \frac{2mb^2}{r^3}\right)$$

$$V(r) = \frac{1}{2}E^2\left(\frac{b^2}{r^2} - \frac{2mb^2}{r^3}\right)$$

$$\frac{dV(r)}{dr} = E^2\left(-\frac{b^2}{r^3} + \frac{3mb^2}{r^4}\right)$$

$$= 0 \quad \text{at} \quad r = 3m.$$

The potential thus has the form



Height of barrier at $r = 3m$ is $V\big|_{r=3m} = \left(\frac{b^2}{27m^2}\right)\frac{1}{2}E^2$.

For photon to fall in, it must just make it over the barrier at $r = 3m$ $\Rightarrow$

$$\left(\frac{dr}{d\lambda}\right)^2\Bigg|_{r=3m} > 0 \iff \left(1 - \frac{b^2}{r^2} + \frac{2mb^2}{r^3}\right)\Bigg|_{r=3m} > 0$$

$$\iff \frac{b^2}{27m^2} < 1 \iff b^2 < 27m^2$$

c) *Near* $r = 3m$:

$$V(r) \approx V(r)\Big|_{r=3m} + \frac{dV}{dr}\Big|_{r=3m}(r-3m) + \frac{1}{2}\frac{d^2V}{dr^2}\Big|_{r=3m}(r-3m)^2$$

$$= \frac{1}{2}E^2\left[\frac{b^2}{27m^2} - \frac{b^2}{81m^4}(r-3m)^2\right]$$

Then

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{2}E^2\left[\left(1-\frac{b^2}{27m^2}\right) + \frac{b^2}{81m^4}(r-3m)^2\right]$$

Let $x = \frac{1}{m}(r-3m)$, $\varepsilon^2 \equiv 3\left(\frac{b^2}{27m^2} - 1\right)$
and assume both are small. Then

$$\left(\frac{dx}{d\lambda}\right)^2 = \frac{1}{6}m^2E^2\left[x^2 - \varepsilon^2\right]$$

The motion comes to a halt at $x = \varepsilon$.
The parameter time taken for this is :

$$\therefore \lambda = \frac{\sqrt{6}}{mE}\int_{x_0}^{\varepsilon}\frac{dx}{(x^2-\varepsilon^2)^{1/2}}$$

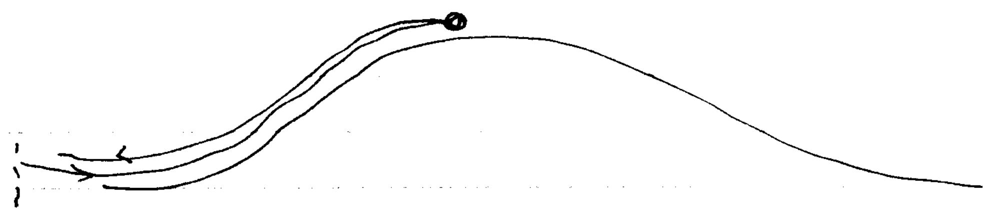
$$= \frac{\sqrt{6}}{2mE}\log\left(\frac{x_0 + \sqrt{x_0^2-\varepsilon^2}}{x_0 - \sqrt{x_0^2-\varepsilon^2}}\right)$$

$$= \frac{\sqrt{6}}{2mE}\log\left(\frac{1+\sqrt{1+(\varepsilon/x_0)^2}}{1-\sqrt{1-(\varepsilon/x_0)^2}}\right)$$

$$\sim -\frac{\sqrt{6}}{mE}\log(\varepsilon) \quad \text{as} \quad \varepsilon \to 0$$

This is log divergent as $\varepsilon \to 0$.

It is a celebrated fact that it takes an arbitrarily long time to roll a ball arbitrarily close to the top of a hill



Time take for trip to within ϵ of top of hill diverges as $-\log(\epsilon)$.

In the mean time, while λ elapses a large ~~number~~ amount ---

$$\left(\frac{d\phi}{d\lambda}\right) = \frac{L}{r^2} \simeq \frac{bE}{(3m)^2}$$

is a finite number --- thus ϕ changes by a large amount -- indeed one can arrange for the photon to stay near $r = 3m$ for as much of a lapse in λ as one wants $\Rightarrow$ one can let ϕ change by as large an amount as once wants --- many, many multiples of 2π if one so chooses.

This means that the observer could see multiple images of the same star --- due to multiple orbits. Except the higher order images will be very, very dim as very few photons will have exactly the correct b to have multiple orbits before coming away from the black hole.

3. Radial motion of particles : $L = 0$

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - 2V(r)$$
$$= (E^2 - 1) + \frac{2m}{r}$$

Starts from rest at $r = R$

$$0 = \left(\frac{dr}{d\tau}\right)^2 = (E^2 - 1) + \frac{2m}{R} \Rightarrow (E^2 - 1) = -\frac{2m}{R}$$

$$\left(\frac{dr}{d\tau}\right)^2 = 2m\left(\frac{1}{r} - \frac{1}{R}\right)$$

Try

$$r = \frac{R}{2}(1 + \cos\eta) \Leftrightarrow \cos\eta = \left(\frac{2r}{R} - 1\right)$$
$$\tau = \frac{R}{2}\left(\frac{R}{2m}\right)^{1/2}(\eta + \sin\eta)$$

$$\left(\frac{dr}{d\tau}\right) = -\frac{R}{2}\sin\eta\left(\frac{d\eta}{d\tau}\right)$$

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{R^2}{4}\sin^2\eta\left(\frac{d\eta}{d\tau}\right)^2$$
$$= \frac{R^2}{4}(1 - \cos^2\eta)\left(\frac{d\eta}{d\tau}\right)^2$$
$$= \frac{R^2}{4}\left(1 - \left(\frac{2r}{R} - 1\right)^2\right)\left(\frac{d\eta}{d\tau}\right)^2$$
$$= (rR - r^2)\left(\frac{d\eta}{d\tau}\right)^2$$

However,

$$\frac{d\tau}{d\eta} = \left(\frac{R}{2}\right)\left(\frac{R}{2m}\right)^{1/2}(1+\cos\eta)$$

$$= \left(\frac{R}{2m}\right)^{1/2} r$$

$$\Rightarrow \left(\frac{d\eta}{d\tau}\right)^2 = \frac{1}{r^2}\left(\frac{2m}{R}\right)$$

$$\Rightarrow \left(\frac{dr}{d\tau}\right)^2 = 2m\left(\frac{1}{r} - \frac{1}{R}\right)$$

which agrees.

From.

$$r = \frac{R}{2}(1+\cos\eta)$$

we see that $\eta = 0 \iff r = R$

$\eta = \pi \iff r = 0$

Moreover, $\tau|_{\eta=0} = 0$

$$\tau|_{\eta=\pi} = \left(\frac{R}{2}\right)\left(\frac{R}{2m}\right)^{1/2}\pi$$

— a finite proper time has elapsed.

For the observer at infinity

$$\frac{dt}{d\tau} = \frac{E}{\left(1 - \frac{2m}{r}\right)}$$

$$\left(\frac{dt}{dr}\right) = -\left(\frac{rR}{2m}\right)^{1/2}\frac{1}{(r-R)^{1/2}}$$

$$\Rightarrow \quad \frac{dt}{dr} = -E\left(\frac{rR}{2m}\right)^{1/2}\frac{r}{(r-R)^{1/2}(r-2m)}$$

$$\Rightarrow \quad t = -\text{const} \times \int \frac{dr}{(r-R)^{1/2}(r-2m)}$$

which is log divergent at $r = 2m$.

I essentially answered the rest of the conundrum in class.

A test particle does take an infinite amount of proper time to reach the horizon... but a test particle has no mass & no gravity of its own. A real particle has mass & distorts the geometry — it falls in in a finite time as seen from infinity because it distorts the geometry of the black hole and its event horizon. The smaller the mass of the infalling particle, the longer it takes as seen from infinity, but real particles never take infinite time as seen from infinity.

4.

For the particle inside the horizon:

$$d\tau^2 = -\left(\frac{2m}{r} - 1\right) du^2 - 2\,du\,dr$$
$$- r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

We want to maximize this quadratic form ... so complete the square:

$$d\tau^2 = -\left[\left(1 - \frac{2m}{r}\right)^{1/2} du + \left(1 - \frac{2m}{r}\right)^{-1/2} dr\right]^2$$
$$+ \left(1 - \frac{2m}{r}\right) dr^2$$
$$- r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

There are three negative terms and only one positive term. To maximize $d\tau^2$ we must set all the negative ones to zero:

$$d\theta = d\phi = 0 \quad \underline{\underline{\text{and}}}$$

$$\left(1 - \frac{2m}{r}\right)^{1/2} du + \left(1 - \frac{2m}{r}\right)^{-1/2} dr = 0$$

$$\Rightarrow \quad \frac{du}{dr} = \left(1 - \frac{2m}{r}\right)^{-1}$$

$$\Rightarrow \quad u = \int \left(\frac{r}{r - 2m}\right) dr$$

$$= \int \left[1 + \frac{2m}{r - 2m}\right] dr$$

$$= \tilde{u}_0 + r + 2m \ln (r - 2m)$$

Thus

$$\theta = \theta_0 \quad , \phi = \phi_0 \quad , u = u_0 + r + 2m \ln\left(1 - \frac{r}{2m}\right)$$

Using the result of question 3:

take $R = 2m$, and then

$$\tau = \frac{\pi R}{2}, = \pi m$$

$$= \left(\frac{Gm\pi}{c^3}\right)$$

$$\approx \frac{6.67 \times 10^{-11} \times 2 \times 10^{30} \times 3.14159}{(3 \times 10^8)^3}$$

$$\approx 1.5 \times 10^{-5} \text{ seconds}$$

We start by taking the metric to be:

$$ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$\Rightarrow \quad g_{00} = -e^{2\alpha}, \quad g^{00} = -e^{-2\alpha}$$

$$g_{11} = e^{2\beta}, \quad g^{11} = e^{-2\beta}$$

I have normalize $F_{\mu\nu}$ different than I did in lectures.

a) $$T_{\mu\nu} = \frac{1}{4\pi}\left\{F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right\}$$

$$F_{01} = f \quad \Rightarrow \quad F_{\rho\sigma}F^{\rho\sigma} = 2g^{00}g^{11}(F_{01})^2$$

$$= -2e^{-2(\alpha+\beta)}f^2$$

$$T_{00} = \frac{1}{4\pi}\left\{g^{11}F_{01}F_{01} - \frac{1}{4}g_{00}\left(-2e^{-2(\alpha+\beta)}f^2\right)\right\}$$

$$= \frac{1}{8\pi}e^{-2\beta}f^2$$

$$T_{11} = \frac{1}{4\pi}\left\{g^{00}F_{10}F_{10} - \frac{1}{4}g_{11}\left(-2e^{-2(\alpha+\beta)}f^2\right)\right\}$$

$$= -\frac{1}{8\pi}e^{-2\alpha}f^2$$

$$T_{22} = \frac{1}{4\pi}\left\{0 - \frac{1}{4}g_{22}\left(-2e^{-2(\alpha+\beta)}f^2\right)\right\}$$

$$= \frac{1}{8\pi}r^2e^{-2(\alpha+\beta)}f^2$$

$$T_{33} = T_{22}\sin^2\theta.$$

$$T = g^{\mu\nu} T_{\mu\nu}$$

$$= \frac{1}{8\pi} \{ -e^{-2\alpha} (e^{-2\beta} f^2) + e^{-2\beta} (- e^{-2\alpha} f^2) + \frac{1}{r^2} (r^2 e^{-2(\alpha+\beta)} f^2) + \frac{1}{r^2 \sin^2\theta} (r^2 e^{-2(\alpha+\beta)} \sin^2\theta f^2) \}$$

$$= 0.$$

b) $$R_{\mu\nu} = 8\pi (T_{\mu\nu} - \tfrac{1}{2} T g_{\mu\nu})$$

$$= 8\pi T_{\mu\nu}$$

From lecture notes, we have the components of $R_{\mu\nu}$

$$R_{00} = -[\beta_{tt} + (\beta_t)^2 - \alpha_t \beta_t] + e^{2(\alpha-\beta)} \times [\alpha_{rr} + (\alpha_r)^2 - \alpha_r \beta_r + \tfrac{2}{r} \alpha_r]$$

$$= e^{-2\beta} f^2$$

$$R_{01} = \frac{2}{r} \beta_t = 0$$

$$R_{11} + e^{2(\beta-\alpha)} R_{00} = \frac{2}{r} (\alpha_r + \beta_r)$$

$$= 8\pi (T_{11} + e^{2(\beta-\alpha)} T_{00}) = e^{-2\alpha} (-f^2 + f^2)$$

$$= 0$$

$$R_{22} = e^{-2\beta} [r(\beta_r - \alpha_r) - 1] + 1 = r^2 e^{-2(\alpha+\beta)} f^2$$

$R_{33} = R_{22} \sin^2\theta = 8\pi T_{33} = 8\pi T_{22} \sin^2\theta.$ (same as above)

Thus we get

$$\beta_t = 0 \quad \Rightarrow \quad \beta \text{ is independent of } t \Rightarrow \beta = \beta(r)$$

$$\alpha_r + \beta_r = 0 \quad \Rightarrow \quad \alpha_r = -\beta_r$$

$$e^{2\alpha}\left[\alpha_{rr} + (\alpha_r)^2 - \alpha_r\beta_r + \frac{2}{r}\alpha_r\right] = f^2$$

$$e^{-2\beta}\left[r(\beta_r - \alpha_r) - 1\right] + 1 = r^2 e^{-2(\alpha+\beta)} \cdot f^2$$

Since $\alpha_r = -\beta_r \Rightarrow$

$$\alpha(r,t) = -\beta(r) + g(t)$$

for some $g(t)$

We apply the reparametrization: $d\tilde{t} = dt\, e^{g(t)}$ to absorb the function $g(t)$. Thus:

$$\alpha = -\beta = \alpha(r).$$

with residual Einstein eq^{n}s.

$$e^{2\alpha}\left[\alpha_{rr} + 2(\alpha_r)^2 + \frac{2}{r}\alpha_r\right] = f^2$$

$$-e^{2\alpha}\left[2r\alpha_r + 1\right] + 1 = r^2 f^2.$$

d) Maxwell eq^{n}s:

$$\sqrt{-g} = \left(e^{\alpha+\beta} r^2 \sin\theta\right)$$

$$X^{\nu} \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, F^{\mu\nu}\right)$$

with $F^{01} = g^{00} g^{11} f = -e^{-2(\alpha+\beta)} f$
$= -f.$

$$X^0 = \frac{\partial}{\partial t} f = 0$$

$$X^1 = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 f\right) = 0$$

$$X^2 = \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \; 0\right) = 0$$

$$X^3 = \frac{\text{stuff}}{} \frac{\partial}{\partial \phi}(0) = 0$$

Thus $f(r,t) = f(r)$

and $\frac{\partial}{\partial r}(r^2 f) = 0 \Rightarrow \boxed{f = (q/r^2)}$

(e) $-e^{2\alpha}\left[2\alpha_r + 1\right] + 1 = q^2/r^2$

$$\Rightarrow + \frac{d}{dr}\left(r e^{2\alpha}\right) = 1 - q^2/r^2$$

$$\Rightarrow r e^{2\alpha} = r + q^2/r + \mu \quad \text{for some const. } \mu.$$

$$\Rightarrow e^{2\alpha} = \left(1 + \frac{M}{r} + q^2/r^2\right)$$

Check other Einstein equation

$$\frac{d}{dr}\left(e^{2\alpha}\right) = 2\alpha_r\, e^{2\alpha}$$

$$\frac{d^2}{dr^2}\left(e^{2\alpha}\right) = \left(2\alpha_{rr} + 4(\alpha_r)^2\right)e^{2\alpha}.$$

Other Einstein eq^n is thus

$$\underbrace{\frac{d^2}{dr^2}\left(\frac{1}{2}e^{2\alpha}\right)}_{\shortparallel} + \underbrace{\frac{1}{r}\frac{d}{dr}\left(e^{2\alpha}\right)}_{\shortparallel} = \underset{\shortparallel}{f^2}$$

$$\frac{1}{2}\left(\cancel{2\frac{M}{r^3}} + \frac{6q^2}{r^4}\right) + \frac{1}{r}\left(\cancel{-\frac{M}{r^2}} - \frac{2q^2}{r^3}\right) = \frac{q^2}{r^4}$$

which is correct.

Once again, by taking the limit as $r \to \infty$ and the Newtonian approximation where

$$g_{00} \approx -(1 + 2\phi) \approx -\left(1 + \frac{2m}{r}\right)$$

one gets $M = 2m$

Thus

$$e^{2\alpha} = \left(1 - \frac{2m}{r} + q^2/r^2\right)$$

6. A stationary observer has a 4-velocity:

$$\frac{dx^\mu}{d\tau} \sim K^\mu$$

and one must have $g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1$, and so

$$\frac{dx^\mu}{d\tau} = V^{-1} K^\mu.$$

Therefore

$$a^\mu = \frac{d^2 x^\mu}{d\tau^2} = (V^{-1} K^\rho) \nabla_\rho (V^{-1} K^\mu)$$

$$= V^{-2} K^\rho \nabla_\rho K^\mu - V^{-3} K^\rho K^\mu (\nabla_\rho V)$$

$$= V^{-2} K^\rho \nabla_\rho K^\mu + \underbrace{\tfrac{1}{2} V^{-4} K^\rho K^\mu \nabla_\rho (K^\nu K_\nu)}_{\text{''}\ \underbrace{K^\rho K^\nu \nabla_\rho K_\nu}_{\text{''}\ 0 \text{ by Killing's eq}^n}}$$

$$= V^{-2} K^\rho \nabla_\rho K^\mu$$

However, $\nabla_\mu V = -\frac{1}{2} V^{-1} \nabla_\mu (K^\rho K_\rho)$

$$= -V^{-1} K^\rho \nabla_\mu K_\rho$$

$$= V^{-1} K^\rho \nabla_\rho K_\mu \quad \text{by Killing's eq}^n$$

$$\Rightarrow \nabla_\mu \ln(V) = V^{-2} K^\rho \nabla_\rho K_\mu$$

and so

$$a_\mu = \nabla_\mu \ln(V)$$

b) By construction

$$a_\mu \frac{dx^\mu}{d\tau} = 0$$

because
$$0 = \frac{d}{d\tau}(-1) = \frac{d}{d\tau}\left(g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) = 2 a_\mu \frac{dx^\mu}{d\tau}.$$

Thus the magnitude of the proper acceleration, $a_\mu a^\mu$, is the magnitude of the spatial acceleration measured in the rest frame of the particle. Thus the magnitude of the acceleration in the rest frame of the particle is

$$a = \left[V^{-2} (\nabla_\mu V)(\nabla^\mu V)\right]^{1/2}$$
$$= V^{-1} \left[(\nabla_\mu V)(\nabla^\mu V)\right]^{1/2}$$

and the corresponding force is

$$F = m V^{-1} \left[(\nabla_\mu V)(\nabla^\mu V)\right]^{1/2}$$

where m is the mass of the particle.

Now imagine doing some work on the particle by pulling the string at infinity. The work done is

$$\Delta E_\infty = F_\infty \delta s$$

where δs is the length of string pulled in at infinity — δs = proper length. The energy transmitted to the particle is

$$\Delta E_p = F \delta s$$

since proper length is preserved. However E_p is measured in the rest frame of the particle. Imagine E_p is converted to a photon and sent back to infinity. We have seen that it undergoes a red-shift so that when the photon gets to infinity it has energy

$$\Delta E_{p \to \infty} = V(\Delta E_p)$$

(see (21.12)) and conservation of energy requires that $\Delta E_{p \to \infty} = \Delta E_\infty$. Hence

$$F_\infty \, \delta s = V \cdot F \, \delta s$$

$$\Rightarrow \quad F_\infty = m \left[(\nabla_\mu V)(\nabla^\mu V) \right]^{1/2}$$

c) In Schwarzschild one has:

$$V = \left(1 - \frac{2M}{r}\right)^{1/2}$$

(t, r, θ, ϕ) coords

$$\Rightarrow \quad \nabla_\mu V = -\left(1 - \frac{2M}{r}\right)^{-1/2} \left(\frac{m}{r^2}\right)^2 (0, 1, 0, 0)$$

$$\Rightarrow \quad (\nabla_\mu V)(\nabla_\nu V) g^{\mu\nu} = \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{M}{r^2}\right)^2 (g^{rr})$$

$$= \frac{M^2}{r^4}$$

$$\Rightarrow \quad \frac{F_\infty}{m} = \frac{M}{r^2}$$

On the event horizon, $r = 2M$

$$\Rightarrow \quad \frac{F_\infty}{m} = \frac{1}{4M}.$$