Physics 530: Fall 2004

Homework 5 **Due: Wednesday, December 1st**

The first, and major part of this homework set will concern metrics of the form:

$$ds^2 \;=\; -2\,du\,dv \;+\; a^2(u)\,dx^2 \;+\; b^2(u)\,dy^2\,. \tag{1}$$

The functions, a and b are, as yet, arbitrary functions of u. To help visualize what it represents, it is convenient to make the change of variables:

$$u \;=\; \tfrac{1}{\sqrt{2}}(t-z)\,, \qquad v \;=\; \tfrac{1}{\sqrt{2}}(t+z)\,, \tag{2}$$

in which the metric takes the form:

$$ds^2 \;=\; -dt^2 \;+\; dz^2 \;+\; a^2\big(\tfrac{1}{\sqrt{2}}(t-x)\big)\,dx^2 \;+\; b^2\big(\tfrac{1}{\sqrt{2}}(t-x)\big)\,dy^2\,. \tag{3}$$

This metric thus represents a gravitational disturbance propagating in the positive z direction at the speed of light. The disturbance is uniform in the x and y directions, and is, for this reason sometimes referred to as a plane-fronted gravitational wave. Your task will be to understand some of what this class of metrics encodes.

N.B. While I have given you the metric in the form (3), I recommend performing computations in the coordinates:

$$(x^0, x^1, x^2, x^3) = (u, v, x, y)\,. \tag{4}$$

You may find it helpful to use (2) and (3) to interpret your results. I will use the index conventions of (4). Be careful in using this system since it means:

$$g_{00} = g_{11} = 0\,, \qquad g^{00} = g^{11} = 0\,, \qquad g_{01} = g_{10} = -1\,, \qquad g^{01} = g^{10} = -1\,. \tag{5}$$

Thus raising or lowering the index "0" converts it to a "1" and vice-versa (it also flips the sign of the component).

Notation: Throughout this homework, a prime, $'$, will denote $\frac{d}{du}$.

1. Compute all the non-zero Christoffel symbols in terms of $a(u), b(u)$, $a'(u)$ and $b'(u)$.

2. Based on your results above, and the symmetry of the Riemann tensor, explain why:

$$R^0{}_{\sigma\mu\nu} = 0\,, \qquad R^\rho{}_{1\mu\nu} = R^\rho{}_{\mu 1\nu} = R^\rho{}_{\mu 1\nu} = 0\,.$$

Compute all the non-zero components of the Riemann tensor in terms of $a(u), b(u)$ and their derivatives.

3. Show that for this metric

$$R_{\mu\nu\rho\sigma}\, R^{\mu\nu\rho\sigma} \equiv 0\,.$$

Now explain why *any* invariant made by any set of contractions of **any number** of Riemann tensors must vanish. For example,

$$R_{\mu\nu\rho\sigma}\, R^{\lambda\nu\tau\sigma}\, R^{\mu\xi\kappa\rho}\, R_{\lambda\tau\kappa\xi} \equiv 0\,.$$

4. Write down, by inspection, **three** independent Killing vectors for the metric (1). Call them $K^\mu_{(1)}$, $K^\mu_{(2)}$ and $K^\mu_{(3)}$. Construct three conserved quantities for geodesic motion. Introduce the functions:

$$A(u) \;\equiv\; \int^u a^{-2}(u')\, du'\,, \qquad B(u) \;\equiv\; \int^u b^{-2}(u')\, du'\,. \tag{6}$$

Find the general solution, $(u(\lambda), v(\lambda), x(\lambda), y(\lambda))$, to the complete set of geodesic equations and express your result in terms of $A(u)$ and $B(u)$. Your answer must include eight arbitrary and independent constants: Four initial positions and four initial velocities. Explain why the coordinate, u, can be used as an affine parameter on the geodesics.

5. Consider the special form of the metric with $a(u) = b(u) = (1-\gamma\, u)$ for some constant, γ:

$$ds^2 \;=\; -2\, du\, dv \;+\; (1-\gamma\, u)^2 \left(dx^2 \;+\; dy^2 \right). \tag{7}$$

This metric appears to be very degenerate, and thus the inverse metric appears to be very singular at $u = \frac{1}{\gamma}$.

Use your results of question 2 to show that that this metric is, in fact, flat (the Riemann tensor is indentically zero). Now find new coordinates (U, V, X, Y) that are functions of (u, v, x, y) in which the metric becomes:

$$ds^2 \;=\; -2\, dU\, dV \;+\; dX^2 \;+\; dY^2 \;=\; -dT^2 \;+\; dZ^2 \;+\; dX^2 \;+\; dY^2\,. \tag{8}$$

where

$$U \;=\; \tfrac{1}{\sqrt{2}}(T - Z)\,, \qquad v \;=\; \tfrac{1}{\sqrt{2}}(T + Z)\,. \tag{9}$$

I want you to find these coordinates in two different ways:

(i) Intelligent guesswork: Take

$$U = u\,, \qquad x = (1-\gamma\,u)^{-1}\,X,\,, \qquad y = (1-\gamma\,u)^{-1}\,Y$$

and then figure out what $v(U, V, X, Y)$ must be to make it work.

(ii) Deduction using geodesic coordinates. Observe that the metric (7) has the desired form at the origin $(u, v, x, y) = (0, 0, 0, 0)$. Use your result in question 4 to find all the geodesics that *start at the origin at* $\lambda = 0$, where λ is a *general* affine parameter. (Warning: Do not try to use u as an affine parameter here... keep the parameter, λ general.) Now follow the prescription in the lecture notes to set up geodesic coordinates, with the initial velocities becoming the new coordinates. If you do this correctly, you will precisely recover the coordinates (U, V, X, Y) that you constructed in part (i).

6. Consider, once again, the general metric (1). Calculate the Ricci tensor and impose the vacuum Einstein equations (with $\Lambda = 0$). What are the differential equation(s) for $a(u)$ and $b(u)$?

Now make the change of dependent variable:

$$a(u) \;=\; f(u)\,e^{\beta(u)}\,, \qquad b(u) \;=\; f(u)\,e^{-\beta(u)}\,, \tag{10}$$

for some functions $f(u)$ and $\beta(u)$. Show that the vacuum Einstein equations imply:

$$f''(u) + (\beta'(u))^2\,f(u) \;=\; 0\,. \tag{11}$$

Comment: It is physically significant and (hopefully) interesting to note that $\beta(u)$ may be chosen arbitrarily and then $f(u)$ is determined by this equation. The function $\beta(u)$ determines the shape, or profile, through the the wave.

7. Now take the simple profile:

$$\beta(u) \;=\; \begin{cases} 0 & u \le 0 \\ \alpha u & 0 \le u \le 1 \\ \alpha & u \ge 1\,. \end{cases}\,, \tag{12}$$

where α is a constant. Note that this function is smooth *except* at $u = 0$ and $u = 1$, where it has simple jump discontinuities in its first derivative.

Take $f(u) = 1$ for $u \le 0$ and find $f(u)$ for all values of u. Choose $f(u)$ to be as smooth as possible, that is, require that $f(u)$ be continuous *and* have continuous first derivatives.

Consider the component R_{0202} of the Riemann tensor and compute its δ-function parts.

Comment: The metric has the form (7) in the region $u > 1$, and thus the metric is completely flat for $u < 0$ and $u > 1$. Here you are looking at a simple solitonic pulse. In general, after any localized solitonic gravitational wave passes, the metric will have the form (7) in the flat space after the wave. Also note that in spite of these δ-functions, all curvature invariants of this metric are identically zero (see problem 3).

7. The equation of motion of a particle of charge, e, rest mass, m_0, in a combined electromagnetic and gravitational field is:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = \frac{e}{m_0}\,F^\mu{}_\nu\frac{dx^\nu}{d\tau}\,. \tag{13}$$

where, as usual, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

a) Justify this equation by examining its form in a local inertial frame.

b) Suppose that ξ^μ is a Killing vector for the metric *and* that

$$\mathcal{L}_\xi F = 0\ ,$$

i.e. the Maxwell field has vanishing Lie derivative along ξ^μ. Give an integral of the motion for the charged particle.

Be Careful: You are *not* given that $\mathcal{L}_\xi A = 0$: The Lie derivative of A might generate a gauge transformation:

$$\mathcal{L}_\xi A \;=\; d\,f\,, \tag{14}$$

for some function, f.

c) Show that $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$ is a constant of the motion.

8. Consider the motion of charged particles in a purely electrostatic Reissner-Nordstrom metric with $m > q$ (*i.e.* the black hole has no magnetic monopole charge, and that the singularity lies inside an event horizon). That is, take:

$$ds^2 \;=\; -\Delta\,dt^2 \;+\; \Delta^{-1}\,dr^2 \;+\; r^2\,(d\theta^2 + \sin^2\theta\,d\phi^2)\,, \qquad A \;=\; -\frac{q}{r}\,dt$$

where

$$\Delta \;\equiv\; \Big(1 - \frac{2m}{r} + \frac{q^2}{r^2}\Big)$$

a) Argue that an orbit always lies in a plane. Choose the plane to be $\theta = \pi/2$. Write down two integrals of the motion that are linear in $\frac{dx^\mu}{d\tau}$, and then use the quadratic integral of 7 c) above to show that the radial equation can be reduced to the form

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V(r) = 0 \ .$$

Give the explicit formula for $V(r)$.

b) Suppose that one has a particle of charge, e, rest mass, m_0 that is fixed at a point (r, θ, ϕ) in the Reissner-Nordström metric. Write an expression for its energy, E, as measured from infinity, and show that if e and q have opposite signs then there is a region outside the outer horizon in which $E < 0$. This is an an "effective ergosphere."

c) Imagine lowering the charged particle (of opposite charge to the black hole) on a string to a point, P, at $r = R$ and then releasing it from rest at P so that it falls (radially) into the black hole. If this is done from inside the ergosphere then energy must be gained at infinity (through the string). Compute the δm and δq for the black hole and show that the (outer) event horizon area does not decrease in this process. Show that if the charge is released *at* the outer horizon then one gets the maximum energy output, and that the area of the horizon remains constant.