Physics 530: Fall 2004

Homework 3 **Due: Wednesday, Oct. 19th**

Problems 1 and 2. On the previous problem set I introduced the natural, Lorentzian metric on a hyperbolic surface defined in the following manner.
Let (t, x, y) be the Cartesian coordinates in $\mathbb{R}^3$, with a Minkowski metric:

$$ds^2 = -dt^2 \; + \; dx^2 \; + \; dy^2 \,. \tag{1}$$

Consider the two dimensional hyperbolic surface $\mathcal{H}$:

$$x^2 + y^2 - t^2 = a^2 \,,$$

and introduce the coordinates, (u, ϕ) on this surface by taking

$$x = a \cosh u \, \cos\phi \,, \qquad y = a \, \cosh u \, \sin\phi \,, \qquad t = a \, \sinh u \,.$$

The metric, $g_{\mu\nu}$ on this hypersurface in the coordinates $(x^1, x^2) = (u, \phi)$ is given by:

$$g_{\mu\nu} \;=\; \begin{pmatrix} -a^2 & 0 \\ 0 & a^2 \cosh^2 u \end{pmatrix} . \tag{2}$$

or, equivalently

$$ds^2 \;=\; -a^2 \, du^2 \; + \; \cosh^2 u \, d\phi^2 \,. \tag{3}$$

Recall that the Christoffel symbols are given by:

$$\Gamma^1_{22} = \sinh u \, \cosh u \,, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \tanh u = \frac{\sinh u}{\cosh u} \,.$$

Also recall that the Riemann tensor is given by:

$$R_{\mu\nu\rho\sigma} = a^{-2}(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \,.$$

1. Consider the family of geodesics that have $\phi =$ constant on each of the geodesics. The parameter, ϕ, thus sweeps out the family of geodesics, and the vector $S^\mu = (0, 1)$ in the (u, ϕ) system is therefore the deviation vector for this family. Verify explicitly that

$$\frac{D}{d\tau}\left(\frac{D}{d\tau} S^\mu\right) = -R^\mu{}_{\rho\nu\sigma} T^\rho S^\nu T^\sigma$$

where

$$T^\rho = \frac{dx^\rho}{d\tau}$$

is the tangent vector to the geodesic in question, and τ is the proper time along the geodesic.

2. Solve Killing's equation in the metric

$$ds^2 = -a^2 du^2 + a^2 \cosh^2 u \, d\phi^2$$

and show that there are *three* independent Killing vectors. Write down the associated integrals of the motion for the geodesic equations. Choose an integral of motion that involves $\frac{du}{d\lambda}$ and verify *explicitly* that it is indeed a constant of the geodesic motion by differentiating it with respect to the affine parameter and using the geodesic equation.

3. Consider a general metric, $g_{\mu\nu}$ on an n-manifold, with the associated infinitessimal "length:"

$$ds^2 \;=\; g_{\mu\nu} \, dx^\mu \, dx^\nu \, . \tag{4}$$

An infinitessimal diffeomorphism is given by:

$$x^\mu \to x^\mu \;+\; \epsilon \, K^\mu(x^\rho) \tag{5}$$

where ϵ is a small constant parameter, and $K^\mu(x^\rho)$ is a vector field. Substitute this into the infinitessimal "length" ds^2 of (4), and collect terms of $\mathcal{O}(\epsilon)$ only. Show that ds^2 is invariant under such a diffeomorphism *if and only if* K^μ satisfies Killing's equation.

4. Let V^μ and W^μ both be Killing vectors.

a) Show that

$$\nabla_\rho \nabla_\sigma V_\mu = R^\nu{}_{\rho\sigma\mu} V_\nu$$

b) $[V, W]^\mu$ is also a Killing vector. Hint: You might find it useful to use part a).

5. In a weak gravitational field in *four* dimensions we can take the metric $g_{\mu\nu}$ to have the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon \, h_{\mu\nu}$$

where ε is a small parameter. Working to first order in ε, show that

$$R_{\mu\nu\rho\sigma} \approx \frac{1}{2}\varepsilon\left(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\nu\sigma,\mu\rho} - h_{\mu\rho,\nu\sigma}\right).$$

Notation: Recall that a subscripted comma denotes an ordinary partial derivative, thus

$$V^{\rho}{}_{,\mu\nu} \;\equiv\; \frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}} V^{\rho}.$$

Now define $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}(h_{\rho}{}^{\rho})\eta_{\mu\nu}$ *where all indices are to be raised and lowered using the Minkowski metric* $\eta_{\mu\nu}$. Show that

$$R_{\mu\nu} \approx \frac{1}{2}\,\varepsilon\left(\tilde{h}_{\mu}{}^{\rho}{}_{,\nu\rho} \;-\; \tilde{h}_{\mu\nu,\rho}{}^{\rho} \;+\; \tilde{h}_{\nu}{}^{\rho}{}_{,\mu\rho} \;+\; \frac{1}{2}\,\eta_{\mu\nu}\,\tilde{h}_{\rho}{}^{\rho}{}_{,\sigma}{}^{\sigma}\right)$$

Consider a small coordinate change in Minkowski space:

$$x^{\mu} \to x^{\mu} + \varepsilon\, f^{\nu}(x^{\rho}).$$

Show that this produces a change

$$h_{\mu\nu} \to h_{\mu\nu} - f_{\mu,\nu} - f_{\nu,\mu}.$$

(Note that to first order in ε this will leave $R_{\mu\nu\rho\sigma}$ and $R_{\mu\nu}$ unchanged.)
Deduce that a coordinate change satisfying

$$\eta^{\mu\nu} f^{\rho}{}_{,\mu\nu} = \tilde{h}^{\rho\sigma}{}_{,\sigma}$$

will make $\tilde{h}^{\mu\nu}{}_{,\nu}$ vanish in the new coordinates, and hence show that the equation

$$R_{\mu\nu} = 0$$

implies

$$\eta^{\rho\sigma} h_{\mu\nu,\rho\sigma} = 0$$

Comments: (i) This is the wave equation in a flat background, and you have now shown that Einstein's equations (with vanishing source) reduce to this in the weak field limit.
(ii) The condition

$$\tilde{h}^{\mu\nu}{}_{,\nu} \;\equiv\; \frac{\partial}{\partial x^{\nu}}\tilde{h}^{\mu\nu} \;=\; 0\,,$$

is the gravitational analogue of the Lorentz gauge condition

$$\frac{\partial}{\partial x^{\nu}}\, A^{\nu} \;=\; 0$$

of electromagnetism, which leads to the wave equation

$$\eta^{\rho\sigma} A^{\mu}{}_{,\rho\sigma} = 0$$

for the 4-vector potential.