Physics 530: Fall 2004

Homework 2 **Due: Monday, Oct. 4th**

1. Recall that the connection coefficient, $\Gamma^{\rho}_{\mu\nu}$ has the transformation property:

$$\Gamma^{\rho'}_{\mu'\nu'} = \left(\frac{\partial x^{\rho'}}{\partial x^{\rho}}\right)\left[\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\,\Gamma^{\rho}_{\mu\nu} + \frac{\partial^2 x^{\rho}}{\partial x^{\mu'}\partial x^{\nu'}}\right]. \quad (1)$$

a) Show that this transformation properties implies that the covariant derivative of a contravariant vector:

$$\nabla_{\mu}V^{\nu} = \partial_{\mu}V^{\nu} + \Gamma^{\nu}_{\mu\rho}\,V^{\rho}\ . \quad (2)$$

is a tensor of type $(1,1)$.

b) Show that the difference of any two affine connections:

$$S^{\rho}{}_{\mu\nu} \equiv \widetilde{\Gamma}^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\nu}$$

is a tensor of type $(1,2)$.

c) Let ∇_{μ} be a covariant derivative satisfying properties 1.–5. **(but not necessarily 6. and 7.)**, Using these properties alone, and (2), show that the covariant derivative of a covariant vector is given by:

$$\nabla_{\mu}W_{\nu} = \partial_{\mu}W_{\nu} - \Gamma^{\rho}_{\mu\nu}\,W_{\rho}\ .$$

2. a) Show that any curve, $x^{\mu}(\alpha)$, satisfying

$$\frac{d^2x^{\mu}}{d\alpha^2} + \Gamma^{\mu}_{\rho\sigma}\frac{dx^{\rho}}{d\alpha}\frac{dx^{\sigma}}{d\alpha} = g(\alpha)\frac{dx^{\mu}}{d\alpha}$$

for an arbitrary function $g(\alpha)$, can be reparametrized using a parameter $\lambda(\alpha)$ such that

$$\frac{d^2x^{\mu}}{dx^2} + \Gamma^{\mu}_{\rho\sigma}\frac{dx^{\rho}}{d\lambda}\frac{dx^{\sigma}}{d\lambda} = 0$$

Obtain an expression for λ in terms of α.

b) A conformal, or Weyl transformation of a metric is defined by scaling it by a positive *function*. That is, given a metric, $g_{\mu\nu}$, we can define a conformally related metric by: $\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ where $\Omega = \Omega(x)$ is some function. Obtain the geodesic equations in the metric $\hat{g}_{\mu\nu}$, and show that null geodesics in $\hat{g}_{\mu\nu}$ are also null geodesics in $g_{\mu\nu}$.

3. Let (t, x, y) be the Cartesian coordinates in $\mathbb{R}^3$, with a Minkowski metric:

$$ds^2 = -dt^2 \; + \; dx^2 \; + \; dy^2 \, . \tag{3}$$

Consider the two dimensional hyperbolic surface $\mathcal{H}$:

$$x^2 + y^2 - t^2 = a^2 \, ,$$

and introduce the coordinates, (u, ϕ) on this surface by taking

$$x = a \cosh u \, \cos\phi \, , \qquad y = a \, \cosh u \, \sin\phi \, , \qquad t = a \, \sinh u \, .$$

a) Show that the metric, $g_{\mu\nu}$ on this hypersurface in the coordinates $(x^1, x^2) = (u, \phi)$ is given by the Lorentzian metric:

$$g_{\mu\nu} \;=\; \begin{pmatrix} -a^2 & 0 \\ 0 & a^2 \cosh^2 u \end{pmatrix} \, . \tag{4}$$

b) Compute the Christoffel symbols by *using the general formula, (8.40) in the notes.* Obtain the geodesic equations on this surface using the formula:

$$\frac{d^2 x^\mu}{dx^2} + \Gamma^\mu{}_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 \, .$$

c) Write down "length functional," $\mathcal{L}$, or proper time for a time-like path, and by explicitly varying $\mathcal{L}$ for the metric (4), once again obtain the geodesic equations on $\mathcal{H}$.

d) Deduce that along a geodesic

$$\cosh^2 u \, \frac{d\phi}{d\lambda} \;=\; h \, , \qquad \text{and} \qquad \cosh^2 u \, \left(\frac{du}{d\lambda}\right)^2 \;=\; h^2 - \ell \, \cosh^2 u \, .$$

e) Show that any geodesic either has $\phi =$ constant, *or* obeys the equation

$$\tanh u = \alpha \sin(\phi - \beta)$$

where α, β are constants; *and* $\alpha^2 > 1$ for timelike, $\alpha^2 = 1$ for null and $\alpha^2 < 1$ for spacelike geodesics.

f) Show that geodesic curves are precisely the intersection in $\mathbb{R}^3$ of the surface, $\mathcal{H}$, and a plane through the origin. Sketch typical timelike, space-like and null geodesics on the hyperboloid. Give a further sketch showing the set of geodesics which start at $u = 0$, $\phi = 0$ in which u and ϕ are regarded as coordinates on an infinite strip of width 2π. Show that no two timelike geodesics, both starting from a point (u_0, ϕ_0), will meet again, but that spacelike geodesics may re-cross each other. Show that there are pairs of points in this deSitter space-time which cannot be joined by a geodesic.

Comment: This hyperbolic space is known as two-dimensional de Sitter space (it has a four dimensional analogue). The spacelike surfaces (i.e. those with time-like normals) are $u = \text{const}$ *—and as time* (u) *gets large the spacelike surfaces grow in size (exponentially fast). Such de Sitter space-times are the basis of the inflationary universe.*

4. Using the metric and de Sitter space of problem 3, suppose that a vector V^μ is parallel transported around a circle $u = u_0$, a constant, with ϕ increasing. Given that

$$V^\mu|_{(u_0,\phi=0)} = (\alpha, \beta)\,,$$

find an expression for

$$V^\mu|_{(u_0,\phi=2\pi)}$$

5. Again, using the metric and de Sitter space of problem 3, compute the Riemann and Ricci tensors, and show that they satisfy

$$R_{\mu\nu\rho\sigma} = a^{-2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\,, \qquad \text{and} \qquad R_{\mu\nu} = a^{-2}g_{\mu\nu}\,. \tag{5}$$

6. Consider an n-dimensional metric of the form:

$$ds^2 = \Omega^2(y^k)\Big(\sum_{j=1}^{n}(dy^j)^2\Big)\,, \tag{6}$$

where $\Omega(y^k)$ is a general function of all the y^k.

a) Find expressions for the Riemann and Ricci tensors of this metric.

b) If one takes

$$\Omega \;=\; a\left[1 \;+\; \frac{1}{4}\sum_{j=1}^{n}(y^j)^2\right]^{-1},$$

then this is the standard metric on an n-sphere of radius a. Show that for this choice of Ω the curvature tensors satisfy (5).