Physics 530: Fall 2004

Homework 1 **Due: Monday, Sept. 13th**

1. (i) Given that $T^{\mu}{}_{\nu\rho}$ is a tensor of type $(1,2)$, show that $S_\mu = T^{\rho}{}_{\mu\rho}$ is a covariant vector.

(ii) Show that if A is a 2-form then dA is a 3-form, that is, verify that dA is a tensor. (Do this by explicitly checking the transformation laws.)

(iii) Show that if A is a 2-form then $d(dA) \equiv 0$.

Comment: Parts (i) and (ii) are, of course, true for an arbitrary k-form. I am simply trying to save you some writing by asking you to check it for a 2-form. In doing this exercise you should note that there is nothing special about $k = 2$.

2. Consider the unit sphere in Cartesian coordinates in $\mathbb{R}^3$:

$$S^2 \equiv \Big\{(x^1, x^2, x^3) : \ (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\Big\} . \qquad (1)$$

In lectures I introduced the stereographic coordinate system defined on $S^2 \backslash N$, where N is the North pole (at $x^3 = 1$):

$$(y^1, y^2) \equiv \left(\frac{2\,x^1}{(1-x^3)}\,, \frac{2\,x^2}{(1-x^3)}\right) . \qquad (2)$$

There is also the natural chart of polar coordinates in which

$$(x^1, x^2, x^3) = (\sin\theta\,\cos\phi\,, \sin\theta\,\sin\phi\,, \cos\theta)\,, \qquad (3)$$

for $0 < \theta < \pi$, $-\pi < \phi < \pi$.

(i) Compute, and simplify $y^\mu(\theta, \phi)$.

Comment: You might find it easier to parametrize everything using $\frac{\theta}{2}$ rather than θ.

(ii) Suppose $W_\mu = (0, 1)$ is a covariant vector in the (y^1, y^2) system. What are the components in the (θ, ϕ) system? Suppose $V^\mu = (0, 1)$ is a contravariant vector in the (y^1, y^2) system. What are the components in the (θ, ϕ) system?

(iii) Invert the coordinate change (2) so as to obtain the Cartesian coordinates, (x^1, x^2, x^3), of a point on the sphere in terms of (y^1, y^2). This expression defines a map (an embedding) of S^2 into $\mathbb{R}^3$. Consider the constant covariant vector field $U_a = (1, 0, 0)$ on $\mathbb{R}^3$. Compute its pull-back, U_μ, to S^2 in the (y^1, y^2) system. Where does U_μ vanish identically? Why does U_μ vanish at some points given that

$U_a = (1,0,0)$ never vanishes?

(iv) Consider a map $\phi : S^2 \to S^2$ in which a point (u,v) in the (y^1, y^2) chart is mapped to another point in the same chart via:

$$\phi : S^2 \to S^2 \quad \text{where} \quad \phi : (u,v) \;\to\; \hat{\rho}^{-2}(2\,u\,, 2\,(\rho^2 - 1))\,, \tag{4}$$

with

$$\rho^2 \;\equiv\; \tfrac{1}{4}\big(u^2 + v^2\big)\,, \qquad \hat{\rho}^2 \;\equiv\; \tfrac{1}{4}\big(u^2 + (v+2)^2\big)\,. \tag{5}$$

In particular, this map takes the great circle at $y^1 = 0$ into itself. I claim that the field, U_μ, that you computed in part (iii) is *invariant* under transportation via ϕ, (*i.e.* ϕ is a symmetry of U_μ). The proof in general is an algebraic mess, and so to simplify things **show that on the the great circle at** $y^1 = 0$, the map, ϕ transports the field U_μ into itself.

Comment: This symmetry is far from obvious in the (y^1, y^2) system. Can you figure out why U_μ is obviously symmetric under this map? Think about where you got U_μ from in the first place ..... (This is not a required part of the answer to this question.)

3. The standard Euclidean metric in Cartesian coordinates in $\mathbb{R}^3$ is given by:

$$ds^2 \;=\; (dx^1)^2 \;+\; (dx^2)^2 \;+\; (dx^3)^2\,. \tag{6}$$

(i) Consider new coordinates (r, θ, ϕ), where:

$$(x^1, x^2, x^3) \;=\; (r\,\sin\theta\,\cos\phi\,, r\,\sin\theta\,\sin\phi\,, r\,\cos\theta)\,. \tag{7}$$

Compute the metric in these new coordinates. (Show your working!)

(ii) From your result in part (i), one can infer that the induced metric on the 2-sphere is:

$$ds^2 \;=\; d\theta^2 \;+\; \sin^2\theta\, d\phi^2\,. \tag{8}$$

Now compute the metric on S^2 in the stereographic coordinates (2). You should find that the metric takes what is known as the "conformally flat" form:

$$ds^2 \;=\; \Omega^2\,\big((dy^1)^2 \;+\; (dy^2)^2\big)\,, \tag{9}$$

for some function, $\Omega(y)$. What is Ω?

4. (i) Let A be a k-form. Obtain a simplified expression (not involving ε-symbols) for the $(k-1)$-form $\delta A \equiv \pm(-1)^{kn+n+1} * d * A$. (Hint: Write ε-tensors in terms of $\sqrt{|g|}$ and the $\tilde{\varepsilon}$-tensor densities).

(ii) Write down a simple expression for δdf where f is a function.

(iii) Compute your expression in part (ii) for $\mathbb{R}^3$, but in the spherical polar coordinates (7). That is, use the metric from **3.** part (i).

5. Consider two points P and P' in a Newtonian gravitational field whose potential is ϕ, and let $\Delta\phi = \phi(P') - \phi(P)$. Suppose one has an excited atom capable of emitting a photon of frequency ν. Consider the following two processes.

A. The atom emits a photon of frequency ν at P, the photon passes through P' where its frequency is measured to be ν'.

B. The atom is carried to P' where it emits the photon of frequency ν, and then the unexcited atom is carried back to P.

By calculating the work done compared to the photon energy measured at P' in both processes, show that

$$\frac{\nu'}{\nu} = (1 - \Delta\phi)\,. \tag{10}$$

(Remember: $E = mc^2$, $E = h\nu$.)