1 The Infinite Solenoid

We will consider a semi–infinite solenoid $L$ lying on the negative $x_3$–axis.

(a) Use the expression (Dirac):

$$A(x) = g \int_L \frac{dl' \times (x - x')}{|x - x'|^3},$$

to show that the vector potential is, in Cartesian coordinates:

$$A_1^N = -\frac{x_2}{|x|(|x| + x_3)}g, \quad (1)$$

$$A_2^N = \frac{x_1}{|x|(|x| + x_3)}g, \quad (2)$$

$$A_3^N = 0. \quad (3)$$

**Hints:** (a) DRAW A BIG DIAGRAM! (b) Recall that

$$\int dx \frac{1}{(a^2 + x^2)^{\frac{3}{2}}} = \frac{x}{(a^2 + x^2)^{\frac{3}{2}} a^2}.$$

(2) Show that in spherical polars, $(r, \theta, \phi)$, where $\phi$ is the polar angle, that the configuration above is:

$$A_r^N = 0, \quad A_\phi^N = \frac{g}{r} \frac{1 - \cos \theta}{\sin \theta}, \quad A_\theta^N = 0. \quad (4)$$
Hint: This is an exercise in recalling how to change from Cartesian to spherical polars. If need be, discuss together in class how to transform the components of a vector. You might (collectively) have to rederive it all from scratch. This does not hurt.

(3) Show that the B field you get from the above potentials gives a point charge of $g$. What happens along $\theta = \pi$?

(4) Write analogous expressions for $A_i^S$ in polars.

(5) Draw the diagram I’ll draw on the board on your answer sheet.

Evaluate the upwards B-flux

$$\int B \cdot dS,$$

through the loop for two cases: (a) $\theta < \frac{\pi}{2}$, (b) $\theta > \frac{\pi}{2}$.

*Hint: It is much easier to evaluate the flux through a neighbouring surface better adapted to the symmetry of B.*

(6) Evaluate the quantity

$$\oint A^N \cdot d\ell$$
around the loop in the indicated direction. What should be the relation between it and the
flux computed above? Study the two cases (a) and (b). Is there a difference? This tells us
something nice about the solenoid.

(7) To make contact with our expressions in the lectures, we can transform to the language of
forms. $A$ is a one–form:

$$A = A_i dx^i.$$  \hspace{1cm} (5)

Examine the expressions above and deduce the tangent space basis for spherical polars:

$$dr, \ r \sin \theta d\phi, \ r d\theta.$$  \hspace{1cm} (6)

(8) Hence verify that we can write

$$A^{N(S)} = \mp g(1 \pm \cos \theta) d\phi.$$  \hspace{1cm} (7)

(9) By asking for the difference between the N and S patches of the two–sphere to be a gauge
transformation

$$A^N - A^S = \frac{i}{q} \Omega d\Omega^{-1}, \quad \Omega = e^{i \omega},$$  \hspace{1cm} (8)

deduce (as we did in the lectures) Dirac’s quantisation condition on the electric charge by
requiring the wavefunction to be single–valued as one goes around (say) the equator.

Do (7)–(9) in ordinary vector language if uncomfortable with the form language.